Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 8 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!
1. (3 points) Solve the following differential equation $y' = 5y$ with initial condition $y(0) = R$.

$$y = Re^{5x}$$

2. (14 points)

$$\begin{array}{c|cccccc}
 x & f(x) & g(x) & j(x) & f'(x) & g'(x) & j'(x) \\
-2 & 0 & 1 & -1 & 3 & 2 & 1 \\
-1 & 1 & 3 & 2 & -1 & 3 & 0 \\
0 & 2 & 1 & 1 & 2 & -2 & 2 \\
1 & 3 & 1 & -1 & 0 & 3 & 1 \\
2 & -2 & 2 & 1 & 3 & 0 & 3 \\
3 & -1 & 1 & -1 & 1 & -2 & 2 \\
\end{array}$$

a. (7 pts) $H(x) = f(j(x)) + g(x^2 - 1)$. Find $H'(2)$.

$$H'(x) = f'(j(x)) \cdot j'(x) + g'(x^2 - 1) \cdot 2x$$

$$H'(2) = 0 \cdot 3 + (-2) \cdot 4 = -8$$

b. (7 pts) $F(x) = \frac{g(x)^2}{(x + 1)j(x)}$. Find $F'(0)$.

$$F'(x) = \frac{2g(x)g'(x)(x + 1)j(x) - (j(x) + (x + 1)j'(x))g(x)^2}{(x + 1)^2j(x)^2}$$

$$F'(0) = \frac{2 \cdot 1 \cdot (-2)(1)(1) - [(1 + 1 \cdot 2) \cdot 1}{1 \cdot 1} = -7$$

3. (20 points) Find $y'$.

a. (10 pts) $y = \frac{\sin^4(x)(3x^4 - 2x + 5)^3e^{3x}}{(x + 1)^2(2x - 3)^5}$ using logarithmic differentiation.

$$y' = \left( \frac{\cos(x)}{\sin(x)} + \frac{36x^3}{3x^4 - 2x + 5} - \frac{2}{x + 1} - \frac{10}{2x - 3} \right) \frac{\sin^4(x)(3x^4 - 2x + 5)^3e^{3x}}{(x + 1)^2(2x - 3)^5}$$

b. (10 pts) $y = \sqrt[4]{\tan^2(2x + 1) + 2\cos(3x) - \arcsin(3x^2)}$

$$\frac{1}{4}(\tan^2(2x + 1))^{-3/4} \cdot 2 \tan(2x + 1) \cdot \sec^2(2x + 1) \cdot 2 + \ln(2)2\cos(3x) \cdot (-3 \sin(3x)) - \frac{6x}{\sqrt{1 - 9x^4}}$$

Or see that the original function can be simplified: $y = \sqrt[4]{\tan(2x + 1) + 2\cos(3x) - \arcsin(3x^2)}$.

$$\frac{1}{2}(\tan(2x + 1))^{-1/2} \cdot \sec^2(2x + 1) \cdot 2 + \ln(2)2\cos(3x) \cdot (-3 \sin(3x)) - \frac{6x}{\sqrt{1 - 9x^4}}$$

These are equivalent.
4. (11 points) For the equation \( \sin(xy^2) = 3x^2 + 3y^3 - 3 \) use implicit differentiation to find \( \frac{dy}{dx} \).

\[
\cos(xy^2) \cdot (y^2 + 2xyy') = 6x + 9y^2 y' \\
y' = \frac{6x - \cos(xy^2)y^2}{\cos(xy^2)2xy - 9y^2}
\]

b. (2 pts) Determine \( \frac{dy}{dx} \) at the point (0,1).

\[ y' \big|_{(0,1)} = \frac{1}{9} \]

5. (18 points) Find the antiderivative of the given function.

a. (6 pts) \( h(x) = x^3 - e^{6x} + 4 \cos(x) - \frac{4}{x} \).

\[ \frac{x^4}{4} - \frac{e^{6x}}{6} + 4 \sin(x) - 4 \ln(x) + C \]

b. (6 pts) \( f(x) = \frac{2}{4 + x^2} \).

\[ \arctan \left( \frac{x}{2} \right) + C \]

c. (6 pts) \( g(x) = \frac{2x}{4 + x^2} \).

\[ \ln(4 + x^2) + C \]
6. (16 points) Evaluate the following limits. Only use L’Hôpital’s rule when appropriate. Show your work!!

a. (8 pts) \( \lim_{x \to \infty} \frac{(\ln x)^2}{x} \)

Type: \( \frac{\infty}{\infty} \)

\[
\lim_{x \to \infty} \frac{(\ln x)^2}{x} = \lim_{x \to \infty} \frac{2 \ln(x) \cdot (x^{-1})}{1} = \lim_{x \to \infty} \frac{2 \ln(x)}{x} = \lim_{x \to \infty} \frac{2x^{-1}}{1} = \lim_{x \to \infty} \frac{2}{x} = 0
\]

b. (8 pts) \( \lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x \), for \( k \) a constant.

\[
y = \lim_{x \to \infty} \left(1 + \frac{k}{x}\right)^x
\]

\[
\ln(y) = \lim_{x \to \infty} x \ln \left(1 + \frac{k}{x}\right), \text{ Type } \infty \cdot 0
\]

\[
\ln(y) = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{k}{x}\right)}{\frac{1}{x}}, \text{ Type } 0^0
\]

\[
\ln(y) = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{k}{x}\right)}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-k}{x-2} = \lim_{x \to \infty} \frac{k}{1 + \frac{k}{x}} = k
\]

\[
y = e^k
\]

7. (3 points) Does \( f(x) = \frac{5x^3 + 6x - 1}{2x^3 + 10} \) have a horizontal asymptote? If so, where is it?

\[
\lim_{x \to \infty} \frac{5x^3 + 6x - 1}{2x^3 + 10}, \text{ Type } \frac{\infty}{\infty}
\]

\[
\lim_{x \to \infty} \frac{5x^3 + 6x - 1}{2x^3 + 10} = \lim_{x \to \infty} \frac{15x^2 + 6}{6x^2} = \lim_{x \to \infty} \frac{30x}{12x} = \frac{5}{2}
\]

Yes, the function has a horizontal asymptote at \( y = \frac{5}{2} \).
8. (15 points) A farmer wishes to build a corral for his cows in the shape below (rectangle with semicircles on the ends). She has 1000ft of fencing and she won’t fence-in the straight edge along the water (just bolded sides). What is the maximal area for the corral?

Label Water side $x$ and the top of the rectangle $y$. The objective is the area of the corral. The constraint is the 1000ft of fencing for the perimeter.

$$A = xy + \frac{\pi y^2}{4}$$

$$1000 = x + \pi y$$
Therefore, $x = 1000 - \pi y$ and $A = (1000 - \pi y)y + \frac{\pi y^2}{4}$.

$$A = 1000y - \frac{3\pi}{4}y^2$$

$$A' = 1000 - \frac{6\pi}{4}y$$

Set the derivative to 0.

$$\frac{6\pi}{4}y = 1000 \rightarrow y = \frac{4000}{6\pi}$$

Check to see if this is a max or min.

$$A'' = -\frac{6\pi}{4}$$

Since the second derivative is always negative, then $y = \frac{4000}{6\pi}$ is a local maximum.

Check for global maximums on the interval $[0,318.3]$.

$A(0) = 0$ sq. ft., $A(1000/\pi) = 79,577.47$ sq. ft (this is just a circle), $A(\frac{4000}{6\pi} \approx 212.2) = 106,103$ sq. ft.

The maximal area is 106,103 sq. ft.