1. Find \( \frac{dy}{dx} \) for each of the following.

(a) \( y = x^2 + 2e^x + e^{2x} + \ln 2 + 2 \ln (2x) + \arctan 2 \)

\[
\frac{dy}{dx} = 2x + \left( \ln 2 \right) 2e^x + 2e^{2x} + \frac{1}{2x} \cdot 2 \\
\text{Note that } e^2, \ln 2, \text{ and } \arctan 2 \text{ are constants.}
\]

(b) \( y = \sqrt{x} \cdot \arctan (5x) \)

\[
\frac{dy}{dx} = \frac{1}{2} x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1 + 25x^2}
\]

(c) \( y = \ln(\tan(2\cos(x^2))) \)

\[
\frac{dy}{dx} = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot \ln 2(2\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x
\]

(d) \( y = \frac{x + e^\pi}{\cos 4 + \sin^5(6x)} \)

\[
\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^\pi)(5 \sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2} \cdot \frac{9}{2} \\
\text{Note that } e^\pi \text{ and } \cos 4 \text{ are constants.}
\]

Recall that \( \sin^5(6x) = (\sin(6x))^5 \).

2. Consider the curve defined by \( x^3 + y^3 = \frac{9}{2}xy \) (known as the Folium of Descartes).

(a) Find \( \frac{dy}{dx} \).

Use implicit differentiation.

\[
3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2} y + \frac{9}{2} x \frac{dy}{dx} \\
3y^2 \frac{dy}{dx} - \frac{9}{2} y \frac{dy}{dx} = \frac{9}{2} y - 3x^2 \\
\frac{dy}{dx} \left( 3y^2 - \frac{9}{2} x \right) = \frac{9}{2} y - 3x^2 \\
\frac{dy}{dx} = \frac{\frac{9}{2} y - 3x^2}{3y^2 - \frac{9}{2} x}
\]

(b) Verify that the point \((1,2)\) is on the curve above.

We must check to see if the values \( x = 1 \) and \( y = 2 \) satisfy the equation above.

\[
x^3 + y^3 = \frac{9}{2}xy \\
1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2 \\
9 = 9
\]

Thus, the point \((1,2)\) is on the curve.

(c) Find the equation of the tangent line at the point \((1,2)\).

We want \( y = mx + b \).

\[
m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5} x + b.
\]

Now plug in \( x = 1 \) and \( y = 2 \) to find \( b \).

\[
2 = \frac{4}{5} \cdot 1 + b \Rightarrow 6 = \frac{5}{5} = b
\]

Therefore, we have \( y = \frac{4}{5} x + \frac{6}{5} \).
3. Evaluate the following limits.
Throughout this solution, the symbol $\star$ will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be “$\equiv$” or $L'\neq H$ or $H \equiv$ or “$0/0$” or “has the form ‘$0$’” and so, by L’Hopital’s Rule, is equal to” or something else. The symbol $\nabla$ will serve the same purpose for the indeterminate forms $\infty/\infty$ and $-\infty/\infty$.

(a) $\lim_{x \to 1} \frac{x^3 - 1}{7 - 7x} \star \lim_{x \to 1} \frac{3x^2}{-7} = \frac{3}{-7} = \frac{3}{7}$

(b) $\lim_{x \to 0} \frac{1 - \cos 2x}{3^x} = \frac{0}{1} = 0$  Can’t use (and don’t need) L’Hopital’s Rule!

(c) $\lim_{x \to 0} \frac{1 - \cos 4x}{5x^2} \star \lim_{x \to 0} \frac{4 \sin 4x}{10x} \star \lim_{x \to 0} \frac{16 \cos 4x}{10} = \frac{16}{10} = \frac{8}{5}$

(d) $\lim_{x \to \infty} \frac{x^2}{2^x} \nabla \lim_{x \to \infty} \frac{2x}{\ln 2 \cdot 2^x} \nabla \lim_{x \to \infty} \frac{\ln 2 \cdot 2^x}{\ln 2 \cdot 2^x} = 0$

4. Rewrite $\tan(\arccos x)$ as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 1:10 section may omit this problem.]

Let $\theta = \arccos x$. That is, $\theta$ is the angle whose cosine is $x$.

$$\tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x}$$

5. Consider the function $f(x) = x^4e^x$ with domain all real numbers.

(a) Find the $x$-value(s) of all roots (x-intercepts) of $f$.
The equation $x^4e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

(b) Find the $x$- and $y$-value(s) of all critical points and identify each as a local max, local min, or neither.

$$f'(x) = 4x^3e^x + x^4e^x$$

$0 = x^3e^x(4 + x)$

$\Rightarrow x = 0, -4$

Note that $e^x$ is never 0.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -4$</td>
<td>positive</td>
</tr>
<tr>
<td>$-4 &lt; x &lt; 0$</td>
<td>negative</td>
</tr>
<tr>
<td>$4 &lt; x$</td>
<td>positive</td>
</tr>
</tbody>
</table>

$y$-values: $f(-4) = 256e^{-4} \approx 4.689, f(0) = 0$

So, $f$ has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

(c) Find the $x$- and $y$-value(s) of all global extrema and identify each as a global max or global min.
There is a global minimum at $(0, 0)$. There is no global maximum because as $x \to \infty, f(x) \to \infty$. Note that as $x \to -\infty, f(x) \to 0$. You can verify this by using L’Hopital’s Rule on $x^4/e^{-x}$.
(d) Find the x-value(s) of all inflection points.

\[ f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \]

Use Product Rule on each product in \( f'(x) \) above.

\[ 0 = e^x(x^4 + 8x^3 + 12x^2) \]
\[ 0 = e^x(x^2 + 8x + 12) \]
\[ 0 = e^x(x^2 + 2)(x + 6) \]
\[ \Rightarrow x = 0, -2, -6 \]

\[
\begin{array}{c|c|c|c|c}
 x & -6 < x < -2 & -2 < x < 0 & 0 < x \\
 f'' & positive & negative & positive & positive \\
 f & concave up & concave down & concave up & concave up \\
\end{array}
\]

So, the x-values of the inflection points of \( f \) are \( x = -2 \) and \( x = -6 \) but NOT \( x = 0 \).

(e) Sketch \( f \).

6. How would your answers to the previous question change if the domain of \( f \) were \([-10, 10]\)?

There would be a global maximum at \((10, 10^4 e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

7. Use Newton’s Method with an initial guess of \( x_0 = -1 \) to find the next three approximations to a solution of \( x^3 + x + 1 = 0 \). Then test your final approximation to see if it appears to be close to a root. [Students in the 8:00, 9:30, and 1:10 sections may omit this problem.]

Recall that \( x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n + 1}{3x_n^2 + 1} \).

Using a calculator with initial guess \( x_0 = -1 \), we get the following.

\[
\begin{align*}
x_0 &= -1 \\
x_1 &= -0.75 \\
x_2 &= -0.68604... \\
x_3 &= -0.68233...
\end{align*}
\]

Check: \((-0.68233)^3 + (-0.68233) + 1 \approx 0.0000526\), so our approximation seems to be a good one.

8. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.
Goal: Maximize volume

Objective function: volume $= V = x \cdot x \cdot y = x^2 y$

We need to get this down to a function of just one variable, so we use the constraint equation:

\[
\text{total cost } = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides})
\]

\[
288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy
\]

\[
288 = 27xy + 24x^2
\]

\[
288 - 24x^2 = 27xy
\]

\[
\frac{288 - 24x^2}{27x} = y
\]

Substituting this back into the objective function gives

\[
V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).
\]

Now that we have $V$ as a function of just one variable, we find its maximum.

\[
V'(x) = \frac{1}{27}(288 - 72x^2)
\]

\[
0 = \frac{1}{27}(288 - 72x^2)
\]

\[
0 = (288 - 72x^2)
\]

\[
72x^2 = 288
\]

\[
x^2 = \frac{288}{72}
\]

\[
x = 2
\]

We discard $x = -2$ because lengths must be nonnegative.

Since $V'$ is positive for $x < 2$ and negative for $2 < x$, we know that the maximum occurs at $x = 2$.

And $y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}$, so the dimensions are $2$ by $2$ by $\frac{32}{9}$. 