\[ \begin{bmatrix} A | I \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 & 9 & -37 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -4 & 12 & 0 & 1 & 0 & 0 \\ -2 & 4 & 2 & -11 & 40 & 0 & 0 & 1 & 0 \\ 4 & 0 & 20 & -7 & 10 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\text{ROW EQUIV:} \quad A

\text{(RREF:)} \quad \begin{bmatrix} 1 & 0 & 5 & 0 & -1 & 0 & -14/3 & 7/6 & 5/6 \\ 0 & 1 & 3 & 0 & 4 & 0 & -29/3 & 8/3 & 4/3 \\ 0 & 0 & 0 & 1 & -2 & 0 & -8/3 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{bmatrix}

\[ \begin{bmatrix} B | I \end{bmatrix} = \begin{bmatrix} -37 & -4 & 3 & 3 & 9 \\ 12 & 1 & 0 & 3 & -4 \\ 40 & 4 & -2 & 2 & -11 \\ 10 & 0 & 4 & 20 & -7 \end{bmatrix} \]

\text{ROW EQUIV:} \quad B

\text{(RREF:)} \quad \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 & 0 & 4/3 & -1/3 & -1/6 \\ 0 & 1 & 0 & 3 & -2 & 0 & -15 & 4 & 2 \\ 0 & 0 & 1 & 5 & -1/2 & 0 & -10/3 & 5/6 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2/3 & 7/6 & -1/6 \end{bmatrix}

1) be neat
2) show your work
3) read the questions
4) Good luck!
1. Let $A$ be as the matrix given on the front cover of this exam; note the RREF form of $[A|I_4]$ is given there.

**1A.** Let $b$ be a column vector in $\mathbb{R}^4$ with entries $b_1, b_2, b_3,$ and $b_4$. What conditions, if any, are there on the $b_i$'s so that $b$ is in Col$(A)$?

The RREF of $[A|I]$ shows $b \in$ Col$(A) \iff \begin{align*} 0 = b_1 - \frac{1}{3} b_2 + \frac{3}{2} b_3 - \frac{1}{2} b_4 \end{align*}$

**1B.** Find a basis for Col$(A)$ using the ideas presented in class.

The pivot cols in RREF $A$ are cols 1, 2, and 4, so the corresponding columns of $A$ form a basis of Col$(A)$. The basis is

$$ \begin{pmatrix} 3 \\ 0 \\ -2 \\ -4 \end{pmatrix}, \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ -4 \\ -11 \end{pmatrix} $$

**1C.** Express the fifth column vector in $A$ as a linear combination of the basis vectors in (1B), and verify that the linear combination is correct by evaluating it; show the work.

The RREF shows easily that col 5 = $-1$ (col 1) + $4$ (col 2) - $2$ col (4).

These same weights show

$$ \begin{bmatrix} 32 \\ 12 \\ 40 \\ 10 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ 0 \\ -2 \\ -4 \end{bmatrix} + 4 \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 9 \\ -4 \\ -11 \end{bmatrix} $$

Let's CHECK: $\begin{bmatrix} -3 \\ -16 \\ -18 \\ 0 + 4 + 8 \\ 2 + 16 + 22 \\ -4 + 0 + 14 \end{bmatrix} = \begin{bmatrix} -37 \\ 12 \\ 40 \end{bmatrix}$ as desired.

**1D.** Find a basis for Nul$(A)$ using the technique presented in class.

The RREF shows solns of $A\vec{x} = 0$ are

$$ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5x_3 + x_4 \\ -3x_3 + 4x_5 \\ x_3 + 0x_5 \\ 0x_3 + 2x_5 \\ 0x_3 + 1x_5 \end{bmatrix} = x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} $$

Here $x_3$ and $x_5$ are FREE and form a basis.

**1E.** Let $B$ be as the matrix given on the front cover of the exam. Note that $B$ has the same columns as $A$, just in a different order. In terms of linear combinations, give a good argument why Col$(B)$ and Col$(A)$ must be identical. (This is a "general truth").

A vector $\tilde{v}$ is in Col$(A) \iff \tilde{v}$ is some L.C. of the columns of $A$. Once such a L.C. is found, the terms can be rearranged (since addition is commutative) to match the order in which $A$'s columns appear in $B$, showing $\tilde{v}$ is a L.C. of the columns of $B$. Any vector in Col$(A)$ is in Col$(B)$ and vice versa, so Col$(A) = \text{Col}(B)$.

**1F.** Is it a coincidence that the last row of RREF form of $[B|I_4]$ is the same as that for $[A|I_4]$? Explain your answer.

No coincidence: since $\text{Col}(A) = \text{Col}(B)$, any conditions $b_1, \ldots, b_4$ must satisfy for $b$ to be in one column space must be the same conditions for $b$ to be in the other. (Here $b = \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix}$.)
2. Again let $B$ be as the matrix given on the front cover of the exam, and suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x) = Bx$.

2A. Find a basis for the kernel of $T$ using the method discussed in class.

We know $\ker(T) = \text{Nul}(B)$. $x \in \text{Nul}(B) \iff x = x_1 \begin{bmatrix} 0 \\ -3 \\ -5 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1/2 \\ -2/5 \\ 1/2 \\ 1 \end{bmatrix}$ where $x_4, x_5$ are free.

Thus a basis is $egin{bmatrix} 0 \\ -3 \\ -5 \\ 0 \end{bmatrix}$.

2B. Find a basis for the image of $T$ (or, range as the book would say) using the theory developed in class.

We know $\text{im}(T) = \text{Col}(B)$ and a basis for the latter is

$$\begin{bmatrix} -3/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3/2 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

2C. Is $T$ one-to-one? If so, explain. If not, give a concrete example showing why not.

$$T \begin{bmatrix} 0 \\ -3/2 \\ -5 \\ 0 \end{bmatrix} = T \begin{bmatrix} 1/2 \\ -2/5 \\ 1/2 \\ 1 \end{bmatrix}$$ (both are $\text{Nul}(T)$ since these two vectors are in the kernel of $T$)

Yet $\begin{bmatrix} 0 \\ -3 \\ -5 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1/2 \\ -2/5 \\ 1/2 \\ 1 \end{bmatrix}$ (so we found two vectors $\tilde{\alpha}, \tilde{\beta}$ s.t. $T(\tilde{\alpha}) = T(\tilde{\beta})$ yet $\tilde{\alpha} \neq \tilde{\beta}$)

there are infinitely many counterexamples, of course!

2D. Is $T$ onto $\mathbb{R}^4$? If so, explain. If not, find a vector not in the image and explain how you found it.

No. There are vectors $b$ in $\mathbb{R}^4$ for which $T(x) = b$ has no solution. Any vector not in $\text{Col}(B)$ will do; we need to find $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ for which $Bx = b$ is inconsistent, and again this happens if $b_1 = -\frac{1}{2}b_2 + \frac{3}{2}b_3 - \frac{3}{2}b_4 \neq 0$.

An easy example is $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

2E. Are $\text{Nul}(B)$ and $\text{Nul}(A)$ equal? Hint: can you express the basis vectors for $\text{Nul}(A)$ as linear combinations of the basis vectors for $\text{Nul}(B)$?

It is obvious that $\begin{bmatrix} -3/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$ (a basis vector of $\text{Nul}(A)$) cannot be expressed as a linear combination of the basis vectors $\begin{bmatrix} 0 \\ -3/2 \\ -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1/2 \\ -2/5 \\ 1/2 \\ 1 \end{bmatrix}$ we found in (2A) for $\text{Nul}(B)$: if you try to set $\begin{bmatrix} -3/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ -3/2 \\ -5 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1/2 \\ -2/5 \\ 1/2 \\ 1 \end{bmatrix}$ the only choice for $\alpha$ and $\beta$ which make the last two rows on both sides equal is $\alpha = 0$ and $\beta = 0$, but then this choice makes the L.S. on the right equal to 0, not (this) $\begin{bmatrix} -3/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$.

So, $\begin{bmatrix} -3/2 \\ -1/2 \\ 0 \\ 0 \end{bmatrix}$ is not in $\text{Nul}(B)$, but of course it is in $\text{Nul}(A)$: $\text{Nul}(A) \neq \text{Nul}(B)$. 


3. Let \( C = \begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \)

3A. Find \( C^{-1} \) by the \([C | I] \sim [I | C^{-1}]\) method we developed in class.

\[
\begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

3B. What elementary matrix \( E \) changes \( C \) into

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

when \( EC \) is computed?

Row 2 of \( C \) has been replaced with Row 2 + 2 · Row 1. Doing this to \( I_4 \) yields \( E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \).

4. Let \( D \) be the matrix

\[
\begin{bmatrix} 13 & 7 & 0 & 15 \\ 11 & -6 & 5 & 8 \\ 0 & 3 & 0 & 6 \\ 1 & 2 & 0 & 4 \end{bmatrix}
\]

4A. Find \( \det(D) \), taking advantage of the 0's in the best possible way. Show all your work!

One "easy path": \( \det(D) = -5 \begin{vmatrix} 13 & 7 & 15 \\ 0 & 3 & 6 \\ 1 & 2 & 4 \end{vmatrix} = -5 \begin{vmatrix} 13 & 3 & 6 \\ 2 & 4 & 7 \\ 1 & 3 & 6 \end{vmatrix} = -5 \begin{vmatrix} 13 & 3 & 6 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = -5 \cdot -3 = 15 \)

4B. Find \( \det(DD) \).

\[
\det(D) \cdot \det(D) = 15 \cdot 15 = 225
\]

4C. Find \( \det(D^T) \).

\[
\det(D^T) = \det(D) = 15
\]

4D. Find \( \det(D^{-1}) \).

\[
\frac{1}{\det(D)} = \frac{1}{15}
\]
5. Let $S$ be the vector spaces of sequences of real numbers as we discussed in class, so

$$ S = \{s = (s_1, s_2, s_3, \ldots) \mid \text{each } s_i \text{ is a real number} \}. $$

Define $T : S \to S$ by $T(s) = (s_1^2, s_2^2, s_3^2, \ldots)$.

5A. Find $T((1, 3, 5, 7, \ldots))$. (write out the first four members of the new sequence).

$$ = (1, 9, 25, 49, \ldots) $$

5B. Show that $T$ is not a linear transformation by using a “concrete” counterexample.

we will show $T(\vec{v}_1 + \vec{v}_2) \neq T(\vec{v}_1) + T(\vec{v}_2)$ in general, let $\vec{v}_1 = (1, 3, 5, 7, \ldots)$ and $\vec{v}_2 = (1, 1, 1, \ldots)$

then $T(\vec{v}_1 + \vec{v}_2) = T((1, 3, 5, 7, \ldots) + (1, 1, 1, \ldots)) = T((2, 4, 6, 8, \ldots)) = (4, 16, 36, 64, \ldots)$

while $T(\vec{v}_1) + T(\vec{v}_2) = T((1, 3, 5, 7, \ldots)) + T((1, 1, 1, \ldots)) = (1, 9, 25, 49, \ldots) + (1, 1, 1, \ldots)$

(NB. there are lots of concrete counterexamples here, of course!)

$$ = (2, 10, 26, 50, \ldots) ; \text{ this is NOT } (4, 16, 36, 64, \ldots) $$

6. Now define $Q : S \to S$ by $Q((s_1, s_2, s_3, \ldots)) = (s_3, s_4, s_5, \ldots)$, so $Q$ just “drops” the original first two elements of the sequence $s$ and shifts all the remaining members of the sequence to the left by two positions. Now $Q$ is indeed a linear transformation; you don’t have to prove it. BUT:

6A. Find $Q((1, 3, 5, 7, \ldots))$. (write out the first four members of the new sequence).

$$ = (5, 7, 9, 11, \ldots) $$

6B. Describe the kernel of $Q$: All sequences $s$ of the form $s = (s_1, s_2, s_3, s_4 \ldots)$ for which ...

$$ \cdots Q((s_1, s_2, s_3, s_4, \ldots)) = \text{ the zero vector, } (0, 0, 0, 0, \ldots) \text{.} $$

Now, because $Q((s_1, s_2, s_3, s_4, \ldots)) = (s_3, s_4, s_5, s_6, \ldots)$ we must have $s_3 = s_4 = s_5 = s_6 = \cdots = 0$ in order for $\vec{s}$ to be in $\text{Ker}(Q)$, but of course the values of $s_1$ and $s_2$ can be ANY real #'.

$$ \left( \text{therefore } \text{Ker}(Q) = \left\{ \vec{s} = (s_1, s_2, \ldots) \in S \mid \vec{s} \text{ is of the form } (s_1, s_2, 0, 0, 0, \ldots) \right\} \right. \text{ where } s_1 \text{ and } s_2 \text{ are arbitrary real #'}s; \text{ Ker } Q \text{ is a 2-0 U.S.} \right) \)
7. Explain why the subset of vectors $H = \left\{ \begin{bmatrix} \sin(\alpha + \beta) \\ \alpha \\ \beta \end{bmatrix} \right\} \in \mathbb{R}^3 \mid \alpha, \beta \in \mathbb{R}$ cannot be closed under vector addition with a concrete counter example. Hint: let $\alpha = \pi/2$ and $\beta = 0$; add the resulting vector to itself. Why isn’t the sum in $H$?

Let $\alpha = \pi/2$ and $\beta = 0$; then $\tilde{v} = \begin{bmatrix} \sin(\pi/2) \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix}$ lie in $H$.

Now $\tilde{v} + \tilde{v} = \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ \pi/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$, which can’t be in $H$ because there are no values of $x$ for which $\sin(x) = 2$ since $-1 \leq \sin x \leq 1$ for all $x \in \mathbb{R}$.

(Alternatively: to write $\begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$ as $\begin{bmatrix} \sin(\alpha' + \beta') \\ \alpha' \\ \beta' \end{bmatrix}$, we must set $\alpha' = \pi$ and $\beta' = 0$.

but then the top element would be $\sin \pi = 0$ instead of 2;

there is no pair $\alpha', \beta'$ for which $\begin{bmatrix} \sin(\alpha' + \beta') \\ \alpha' \\ \beta' \end{bmatrix} = \begin{bmatrix} 2 \\ \pi \\ 0 \end{bmatrix}$)

8. For each of the following vector spaces, give an obvious basis, or explain why there is no basis, and then give the dimension of the vector space:

8A. $\mathbb{R}^3$

\[ \text{(obvious) BASIS} \]
\[ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ \text{DIMENSION} \]
\[ 3 \]

8B. $\{0\}$

NO basis: the only set of vectors in $\{0\}$ which spans this V.S. is $\{0\}$ itself, but this set is not L.I.

\[ \text{DIMENSION} \]
\[ 0 \]

8C. $P_3$

\[ \{ 1, t, t^2, t^3 \} \]

or, maybe $\{ 1, t+1, t^2+t+1, t^3+t^2+t+1 \}$

but NOT $\{ a+bt+ct^2+dt^3 \}$