\[ [A | I] = \begin{bmatrix} 3 & -4 & 3 & 9 & -37 \\ -2 & 1 & 3 & -4 & 12 \\ 4 & 0 & 0 & 20 & -11 & 40 \\ 0 & 0 & 20 & -7 & 10 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \text{ROW EQUIV:} \quad A \]

\[ [B | I] = \begin{bmatrix} -37 & -4 & 3 & 3 & 9 \\ 12 & 1 & 3 & -4 \\ 40 & 4 & -2 & 2 & -11 \\ 10 & 0 & 4 & 20 & -7 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \text{ROW EQUIV:} \quad B \]

\[ \text{(RREF:)} \]

\[ \begin{bmatrix} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 3 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4/3 & -1/3 & -1/6 \end{bmatrix} \]

1) be neat
2) show your work
3) read the questions
4) GOOD LUCK!
1. Let $A$ be as the matrix given on the front cover of this exam; note the RREF form of $[A|I_4]$ is given there.

1A. Let $b$ be a column vector in $\mathbb{R}^4$ with entries $b_1, b_2, b_3,$ and $b_4$. What conditions, if any, are there on the $b_i$’s so that $b$ is in $\text{Col}(A)$?

1B. Find a basis for $\text{Col}(A)$ using the ideas presented in class.

1C. Express the fifth column vector in $A$ as a linear combination of the basis vectors in (1B), and verify that the linear combination is correct by evaluating it; show the work.

1D. Find a basis for $\text{Nul}(A)$ using the technique presented in class.

1E. Let $B$ be as the matrix given on the front cover of the exam. Note that $B$ has the same columns as $A$, just in a different order. In terms of linear combinations, give a good argument why $\text{Col}(B)$ and $\text{Col}(A)$ must be identical. (This is a “general truth”).

1F. Is it a coincidence that the last row of RREF form of $[B|I_4]$ is the same as that for $[A|I_4]$? Explain your answer.
2. Again let $B$ be as the matrix given on the front cover of the exam, and suppose $T : \mathbb{R}^5 \to \mathbb{R}^4$ is defined by $T(x) = Bx$.

2A. Find a basis for the kernel of $T$ using the method discussed in class.

2B. Find a basis for the image of $T$ (or, *range* as the book would say) using the theory developed in class.

2C. Is $T$ one-to-one? If so, explain. If not, give a concrete example showing why not.

2D. Is $T$ onto $\mathbb{R}^4$? If so, explain. If not, find a vector not in the image and explain how you found it.

2E. Are $\text{Nul}(B)$ and $\text{Nul}(A)$ equal? Hint: can you express the basis vectors for $\text{Nul}(A)$ as linear combinations of the basis vectors for $\text{Nul}(B)$?
3. Let \( C = \begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \)

3A. Find \( C^{-1} \) by the \( "[C|I] \sim [I|C^{-1}]" \) method we developed in class.

3B. What elementary matrix \( E \) changes \( C \) into
\[
\begin{bmatrix} 0 & 12 & 4 & -1 \\ 0 & 25 & 8 & -2 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\] when \( EC \) is computed?

4. Let \( D \) be the matrix
\[
\begin{bmatrix} 13 & 7 & 0 & 15 \\ 11 & -6 & 5 & 8 \\ 0 & 3 & 0 & 6 \\ 1 & 2 & 0 & 4 \end{bmatrix}
\]

4A. Find \( \det(D) \), taking advantage of the 0's in the best possible way. Show all your work!

4B. Find \( \det(DD) \).

4C. Find \( \det(D^T) \).

4D. Find \( \det(D^{-1}) \).
5. Let $S$ be the vector spaces of sequences of real numbers as we discussed in class, so $S = \{ s = (s_1, s_2, s_3, \ldots) \mid \text{each } s_i \text{ is a real number} \}$.

Define $T : S \to S$ by $T(s) = (s_1^2, s_2^2, s_3^2, \ldots)$.

5A. Find $T((1, 3, 5, 7, \ldots))$. (write out the first four members of the new sequence).

5B. Show that $T$ is not a linear transformation by using a “concrete” counterexample.

6. Now define $Q : S \to S$ by $Q((s_1, s_2, s_3, \ldots)) = (s_3, s_4, s_5, \ldots)$, so $Q$ just “drops” the original first two elements of the sequence $s$ and shifts all the remaining members of the sequence to the left by two positions. Now $Q$ is indeed a linear transformation; you don’t have to prove it. BUT:

6A. Find $Q((1, 3, 5, 7, \ldots))$. (write out the first four members of the new sequence).

6B. Describe the kernel of $Q$: All sequences $s$ of the form $s = (s_1, s_2, s_3, s_4 \ldots)$ for which \ldots
7. Explain why the subset of vectors \( H = \left\{ \begin{bmatrix} \sin(\alpha + \beta) \\ \alpha \\ \beta \end{bmatrix} \in \mathbb{R}^3 \mid \alpha, \beta \in \mathbb{R} \right\} \) cannot be closed under vector addition with a concrete counter example. Hint: let \( \alpha = \pi/2 \) and \( \beta = 0 \); add the resulting vector to itself. Why isn’t the sum in \( H \)?

8. For each of the following vector spaces, give an obvious basis, or explain why there is no basis, and then give the dimension of the vector space:

8A. \( \mathbb{R}^3 \)

8B. \( \{0\} \)

8C. \( \mathbb{P}_3 \)