Math 105: Review for Exam II - Solutions

1. Find \( \frac{dy}{dx} \) for each of the following.
   (a) \( y = x^2 + 2x^2 + e^x + e^{2x} + \ln(2 + \ln(2x)) + (\ln(2)x) + \arctan(2) \)
   \[
   \frac{dy}{dx} = 2x + (\ln(2)2^x + 2e^{2x} + \left(\frac{1}{2x}\right)2 + \ln(2)
   \]
   Note that \( e^x \), \( \ln(2) \), and \( \arctan(2) \) are constants.
   
   (b) \( y = \sqrt{x} \cdot \arctan(5x) \)
   \[
   \frac{dy}{dx} = \frac{1}{2x^{1/2}} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2} \]
   
   (c) \( y = \ln(\tan(2\cos(x^2))) \)
   \[
   \frac{dy}{dx} = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot \ln(2(2\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x \]
   
   (d) \( y = \frac{x + e^x}{\cos(4 + \sin^5(6x))} \)
   Note that \( e^x \) and \( \cos(4) \) are constants.
   \[
   \frac{dy}{dx} = \frac{(1)(\cos(4 + \sin^5(6x)) - (x + e^x)(5\sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos(4 + \sin^5(6x))^2} \]
   Recall that \( \sin^5(6x) = (\sin(6x))^5 \).
   
   (e) \( y = (x^2 + 1)^\sin x \)
   We need logarithmic differentiation here.
   \[
   \ln y = \sin x \cdot \ln(x^2 + 1) \]
   Take natural log of each side.
   \[
   \frac{1}{y} \frac{dy}{dx} = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x \]
   Differentiate.
   \[
   \frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] y \]
   Solve for \( \frac{dy}{dx} \).
   \[
   \frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] (x^2 + 1)^\sin x \]
   Replace \( y \).

2. Consider the curve defined by \( x^3 + y^3 = \frac{9}{2}xy \) (known as the Folium of Descartes).
   (a) Find \( \frac{dy}{dx} \).
   Use implicit differentiation.
   \[
   3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \]
   \[
   3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} = \frac{9}{2}y - 3x^2 \]
   \[
   \frac{dy}{dx} \left( 3y^2 - \frac{9}{2}x \right) = \frac{9}{2}y - 3x^2 \]
   \[
   \frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \]
   
   (b) Verify that the point \( (1,2) \) is on the curve above.
   We must check to see if the values \( x = 1 \) and \( y = 2 \) satisfy the equation above.
   \[
   x^3 + y^3 = \frac{9}{2}xy \]
   \[
   1^3 + 2^3 \geq \frac{9}{2} \cdot 1 \cdot 2 \]
   \[
   9 \geq 9 \]
   Thus, the point \( (1,2) \) is on the curve.
(c) Find the equation of the tangent line at the point (1,2).

We want \( y = mx + b \).

\[
m = \frac{\frac{9}{3} \cdot 2 - 3 \cdot 1^2}{2 \cdot 2 - \frac{2}{1} \cdot 1} = \frac{4}{5}, 
\]

so \( y = \frac{4}{5} x + b \).

Now plug in \( x = 1 \) and \( y = 2 \) to find \( b \).

\[
2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b
\]

Therefore, we have \( y = \frac{4}{5} x + \frac{6}{5} \).

3. Evaluate the following limits.

Throughout this solution, the symbol \( \star \) will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be \( \frac{0}{0} \), \( \frac{\infty}{\infty} \), \( \frac{\infty}{0} \) or “has the form \( \frac{1}{0} \)”, and so, by L’Hopital’s Rule, is equal to” or something else. The symbol \( \heartsuit \) will serve the same purpose for the indeterminate forms \( \frac{\infty}{\infty} \) and \( \frac{-\infty}{\infty} \).

(a) \( \lim_{x \to 1} \frac{x^3 - 1}{7 - 7x} \)

\[
\lim_{x \to 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}
\]

(b) \( \lim_{x \to 0} \frac{1 - \cos(2x)}{3x} \)

Cannot use (and don’t need) L’Hopital’s Rule!

(c) \( \lim_{x \to 0} \frac{1 - \cos(4x)}{5x^2} \)

\[
\lim_{x \to 0} \frac{4 \sin(4x)}{10x} \quad \lim_{x \to 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5}
\]

(d) \( \lim_{x \to \infty} \frac{x^2}{2x} \quad \lim_{x \to \infty} \frac{2x}{\ln 2 \cdot 2^x} \quad \lim_{x \to \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0
\]

(e) \( \lim_{x \to \infty} (1 + \frac{4}{x})^{\frac{503x}{x}} \)

[Students in the 8:00 and 9:30 sections may omit this problem.]

Let \( L = \lim_{x \to \infty} (1 + \frac{4}{x})^{\frac{503x}{x}} \).

This is of the form \( 1^\infty \), so we take the ln of each side to turn it into a L’Hopital’s Rule problem.

\[
\ln L = \lim_{x \to \infty} \ln \left(1 + \frac{4}{x}\right)^{\frac{503x}{x}} 
\]

\[
\ln L = \lim_{x \to \infty} (503x) \ln \left(1 + \frac{4}{x}\right) 
\]

Apply the log rule \( \ln(a^b) = b \ln(a) \).

\[
\ln L = \lim_{x \to \infty} \frac{\ln \left(1 + \frac{4}{x}\right)}{\frac{1}{503x}} 
\]

Rewrite so it takes the form \( \frac{0}{0} \).

\[
\ln L = \lim_{x \to \infty} \frac{(1 + \frac{4}{x}) \cdot \frac{4}{x^2}}{\frac{1}{503x}} 
\]

Apply L’Hopital’s Rule.

\[
\ln L = \lim_{x \to \infty} \frac{(1 + \frac{4}{x}) \cdot \frac{4}{503}}{1} 
\]

Cancel terms.

\[
\ln L = \lim_{x \to \infty} \frac{1 + \frac{4}{x}}{\frac{503}{503}} 
\]

Take the limit.

\[
\ln L = \frac{1 + 0}{503} = \frac{4}{503} 
\]

Now, since \( \ln L = \frac{4}{503} = 2012 \), we have \( L = e^{2012} \).
4. Rewrite \( \tan(\arccos x) \) as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 8:00, 9:30, and 1:10 sections may omit this problem.]

Let \( \theta = \arccos x \). That is, \( \theta \) is the angle whose cosine is \( x \).

\[
\begin{align*}
1 & \quad y \\
y & \quad x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1 - x^2}
\end{align*}
\]

\( \tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x} \)

5. Consider the function \( f(x) = x^4 e^x \) with domain all real numbers.

(a) Find the \( x \)-value(s) of all roots (\( x \)-intercepts) of \( f \).

The equation \( x^4 e^x = 0 \) means \( x^4 = 0 \) (that is, \( x = 0 \)) or \( e^x = 0 \) (no solution), so the only root is at \( x = 0 \).

(b) Find the \( x \)- and \( y \)-value(s) of all critical points and identify each as a local max, local min, or neither.

\[
f'(x) = 4x^3 e^x + x^4 e^x
\]

\[
0 = x^3 e^x (4 + x)
\]

\( \Rightarrow x = 0, -4 \) \hspace{1cm} \text{Note that } e^x \text{ is never 0.}

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f' )</th>
<th>( f )</th>
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<tbody>
<tr>
<td>( x &lt; -4 )</td>
<td>positive</td>
<td>concave up</td>
</tr>
<tr>
<td>( -4 &lt; x &lt; 0 )</td>
<td>negative</td>
<td>concave down</td>
</tr>
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\( y \)-values: \( f(-4) = 256e^{-4} \approx 4.689 \), \( f(0) = 0 \)

So, \( f \) has a local maximum at \((-4, 256e^{-4})\) and a local minimum at \((0, 0)\).

(c) Find the \( x \)- and \( y \)-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at \((0, 0)\). There is no global maximum because as \( x \to \infty \), \( f(x) \to \infty \). Note that as \( x \to -\infty \), \( f(x) \to 0 \). You can verify this by using L’Hopital’s Rule on \( x^4/e^{-x} \).

(d) Find the \( x \)-value(s) of all inflection points.

\[
f''(x) = 12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.}
\]

\[
0 = e^x (x^4 + 8x^3 + 12x^2)
\]

\[
0 = e^x x^2 (x^2 + 8x + 12)
\]

\[
0 = e^x x^2 (x + 2)(x + 6)
\]

\( \Rightarrow x = 0, -2, -6 \)

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<tr>
<th>( f' )</th>
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So, the \( x \)-values of the inflection points of \( f \) are \( x = -2 \) and \( x = -6 \) but NOT \( x = 0 \).

(e) Sketch \( f \).
6. How would your answers to the previous question change if the domain of \( f \) were \([-10,10]\)?
   There would be a global maximum at \((10, 10^{4} e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.

   ![Diagram of aquarium]

   **Goal:** Maximize volume

   **Objective function:** volume \( V = x \cdot x \cdot y = x^2 y \)

   We need to get this down to a function of just one variable, so we use the constraint equation:

   \[
   \text{total cost} = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides})
   \]

   \[
   288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy
   \]

   \[
   288 - 24x^2 = 27xy
   \]

   \[
   \frac{288 - 24x^2}{27x} = y
   \]

   Substituting this back into the objective function gives

   \[
   V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27} (288x - 24x^3).
   \]

   Now that we have \( V \) as a function of just one variable, we find its maximum.

   \[
   V'(x) = \frac{1}{27} (288 - 72x^2)
   \]

   \[
   0 = \frac{1}{27} (288 - 72x^2)
   \]

   \[
   0 = (288 - 72x^2)
   \]

   \[
   72x^2 = 288
   \]

   \[
   x^2 = \frac{288}{72}
   \]

   \[
   x = 2
   \]

   We discard \( x = -2 \) because lengths must be nonnegative.

   Since \( V' \) is positive for \( x < 2 \) and negative for \( 2 < x \), we know that the maximum occurs at \( x = 2 \).

   And \( y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9} \), so the dimensions are 2 by 2 by \( \frac{32}{9} \).