Math 105: Review for Exam II - Solutions

1. Find \( \frac{dy}{dx} \) for each of the following.

(a) \( y = x^2 + 2x + e^x + 2e^{2x} + \ln 2 + \ln(2x) + (\ln 2)x + \arctan 2 \)
\[
\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2 + \ln 2 \quad \text{Note that } e^2, \ln 2, \text{ and } \arctan 2 \text{ are constants.}
\]

(b) \( y = \sqrt{x} \cdot \arctan(5x) \)
\[
\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}
\]

(c) \( y = \ln(\tan(2\cos(x^2))) \)
\[
\frac{dy}{dx} = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot \ln(2\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x
\]

(d) \( y = \frac{x + e^x}{\cos 4 + \sin^5(6x)} \)
\[
\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^x)5 \sin^4(6x) \cdot \cos(6x) \cdot 6}{(\cos 4 + \sin^5(6x))^2} \quad \text{Recall that } \sin^5(6x) = (\sin(6x))^5.
\]

(e) \( y = (x^2 + 1)^{\sin x} \)
\[
\ln y = x \cdot \ln(x^2 + 1) \quad \text{Take natural log of each side.}
\]
\[
\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x \quad \text{Differentiate.}
\]
\[
\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] y \quad \text{Solve for } \frac{dy}{dx}.
\]
\[
\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot (x^2 + 1)^{\sin x} \quad \text{Replace } y.
\]

2. Consider the curve defined by \( x^3 + y^3 = \frac{9}{2}xy \) (known as the Folium of Descartes).

(a) Find \( \frac{dy}{dx} \).
\[
3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx}
\]
\[
3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} = \frac{9}{2}y - 3x^2
\]
\[
\frac{dy}{dx} \left( 3y^2 - \frac{9}{2}x \right) = \frac{9}{2}y - 3x^2
\]
\[
\frac{dy}{dx} = \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x}
\]

(b) Verify that the point \((1, 2)\) is on the curve above.
We must check to see if the values \( x = 1 \) and \( y = 2 \) satisfy the equation above.
\[
x^3 + y^3 = \frac{9}{2}xy
\]
\[
1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2
\]
\[
9 = 9
\]

Thus, the point \((1, 2)\) is on the curve.
(c) Find the equation of the tangent line at the point (1,2).

We want \( y = mx + b \).

\[
m = \frac{\frac{2}{3} \cdot 2 - 3 \cdot 1^2}{\frac{2}{3} \cdot 2^2 - \frac{2}{3} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.
\]

Now plug in \( x = 1 \) and \( y = 2 \) to find \( b \).

\[
2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b
\]

Therefore, we have \( y = \frac{4}{5}x + \frac{6}{5} \).

3. Evaluate the following limits.

Throughout this solution, the symbol \( \star \) will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form 0/0; this may be “\( \frac{0}{0} \)” or \( \frac{L}{L} \) or \( \frac{0}{\infty} \) or “has the form \( \frac{0}{0} \)” and so, by L’Hopital’s Rule, is equal to” or something else. The symbol \( \bigtriangledown \) will serve the same purpose for the indeterminate forms \( \infty/\infty \) and \( -\infty/\infty \).

(a) \( \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \star \lim_{x \to 1} \frac{3x^2}{1} = \frac{3}{1} = 3 \)

(b) \( \lim_{x \to 0} \frac{1 - \cos(2x)}{3x^2} = \frac{0}{1} = 0 \)

Can’t use (and don’t need) L’Hopital’s Rule!

(c) \( \lim_{x \to 0} \frac{1 - \cos(4x)}{5x^2} \star \lim_{x \to 0} \frac{4 \sin(4x)}{10x} \star \lim_{x \to 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5} \)

(d) \( \lim_{x \to \infty} \frac{x^2}{2^x} \bigtriangledown \lim_{x \to \infty} \frac{2x}{\ln 2 \cdot 2^x} \bigtriangledown \lim_{x \to \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0 \)

4. Consider the function \( f(x) = x^4e^x \) with domain all real numbers.

(a) Find the \( x \)-value(s) of all roots (x-intercepts) of \( f \).

The equation \( x^4e^x = 0 \) means \( x^4 = 0 \) (that is, \( x = 0 \)) or \( e^x = 0 \) (no solution), so the only root is at \( x = 0 \).

(b) Find the \( x \)- and \( y \)-value(s) of all critical points and identify each as a local max, local min, or neither.

\( f'(x) = 4x^3e^x + x^4e^x \)

\[
0 = x^3e^x(4 + x)
\]

\( \Rightarrow x = 0, -4 \)

Note that \( e^x \) is never 0.

\[
\begin{array}{c|c|c|c}
& x < -4 & -4 < x < 0 & 4 < x \\
\hline
f' & \text{positive} & \text{negative} & \text{positive} \\
\hline
\end{array}
\]

\( y \)-values: \( f(-4) = 256e^{-4} \approx 4.689, f(0) = 0 \)

So, \( f \) has a local maximum at \((-4, 256e^{-4})\) and a local minimum at \((0, 0)\).

(c) Find the \( x \)- and \( y \)-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at \((0, 0)\). There is no global maximum because as \( x \to \infty, f(x) \to \infty \). Note that as \( x \to -\infty, f(x) \to 0 \). You can verify this by using L’Hopital’s Rule on \( x^4/e^{-x} \).
(d) Find the $x$-value(s) of all inflection points.

$$f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x$$

Use Product Rule on each product in $f'(x)$ above.

$$0 = e^x(x^4 + 8x^3 + 12x^2)$$

$$0 = e^x(x^2 + 8x + 12)$$

$$0 = e^x(x + 2)(x + 6)$$

$\Rightarrow x = 0, -2, -6$

<table>
<thead>
<tr>
<th>$f''$</th>
<th>$x &lt; -6$</th>
<th>$-6 &lt; x &lt; -2$</th>
<th>$-2 &lt; x &lt; 0$</th>
<th>$0 &lt; x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td>concave up</td>
<td>concave down</td>
<td>concave up</td>
<td>positive</td>
</tr>
<tr>
<td>$f$</td>
<td>concave up</td>
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<td>concave up</td>
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</tbody>
</table>

So, the $x$-values of the inflection points of $f$ are $x = -2$ and $x = -6$ but NOT $x = 0$.

(e) Sketch $f$.

5. How would your answers to the previous question change if the domain of $f$ were $[-10, 10]$?

There would be a global maximum at $(10, 10^4e^{10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

6. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.

![Diagram of a box-shaped aquarium]

**Goal**: Maximize volume

**Objective function**: $V = x \cdot x \cdot y = x^2y$

We need to get this down to a function of just one variable, so we use the constraint equation:

$$288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy$$

$$288 = 27xy + 24x^2$$

$$288 - 24x^2 = 27xy$$

$$\frac{288 - 24x^2}{27x} = y$$

Substituting this back into the objective function gives
\[ V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27} (288x - 24x^3). \]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[ V'(x) = \frac{1}{27} (288 - 72x^2) \]

\[ 0 = \frac{1}{27} (288 - 72x^2) \]

\[ 0 = (288 - 72x^2) \]

\[ 72x^2 = 288 \]

\[ x^2 = \frac{288}{72} \]

\[ x = 2 \]

We discard \( x = -2 \) because lengths must be nonnegative.

Since \( V' \) is positive for \( x < 2 \) and negative for \( 2 < x \), we know that the maximum occurs at \( x = 2 \).

And \( y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9} \) so the dimensions are 2 by 2 by \( \frac{32}{9} \).