Math 105: Review for Exam II - Solutions

1. Find $dy/dx$ for each of the following.

(a) \[ y = x^2 + 2x + e^x + e^{2x} + \ln 2 + \ln(2x) + (\ln 2)x + \arctan 2 \]
\[ dy \over dx = 2x + (\ln 2)2x + 2e^{2x} + \frac{1}{2x} \cdot 2 + \ln 2 \]
Note that $e^2$, ln 2, and arctan 2 are constants.

(b) \[ y = \sqrt{x} \cdot \arctan(5x) \]
\[ dy \over dx = \frac{1}{2x} \cdot \arctan(5x) + \sqrt{x} \cdot \frac{1}{1 + (5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1 + 25x^2} \]

(c) \[ y = \ln(\tan(2\cos(x^2))) \]
\[ dy \over dx = \frac{1}{\tan(2\cos(x^2))} \cdot \sec^2(2\cos(x^2)) \cdot \ln(2\cos(x^2)) \cdot (-\sin(x^2)) \cdot 2x \]

(d) \[ y = \frac{x + e^x}{\cos 4 + \sin^5(6x)} \]
\[ dy \over dx = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^x)(5\sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2} \]
Recall that $\sin^5(6x) = (\sin(6x))^5$.

(e) \[ y = (x^2 + 1)^{\sin x} \]
We need logarithmic differentiation here.
\[ \ln y = \sin x \cdot \ln(x^2 + 1) \]
\[ 1 \over y \cdot dy \over dx = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x \]
Differentiate.
\[ {dy \over dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] y \]
Solve for $dy \over dx$.
\[ {dy \over dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot (x^2 + 1)^{\sin x} \]
Replace $y$.

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) Find $dy/dx$. Use implicit differentiation.
\[ 3x^2 + 3y^2 \frac{dy}{dx} = \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \]
\[ 3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} = \frac{9}{2}y - 3x^2 \]
\[ \frac{dy}{dx} \left( 3y^2 - \frac{9}{2}x \right) = \frac{9}{2}y - 3x^2 \]
\[ \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - \frac{9}{2}x} \]

(b) Verify that the point $(1,2)$ is on the curve above.
We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.
\[ x^3 + y^3 = \frac{9}{2}xy \]
\[ 1^3 + 2^3 = \frac{9}{2} \cdot 1 \cdot 2 \]
\[ 9 = 9 \]
Thus, the point $(1,2)$ is on the curve.
(c) Find the equation of the tangent line at the point (1,2).

We want \( y = mx + b \).

\[
m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{5}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.
\]

Now plug in \( x = 1 \) and \( y = 2 \) to find \( b \).

\[
2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b
\]

Therefore, we have \( y = \frac{4}{5}x + \frac{6}{5} \).

3. Evaluate the following limits.

Throughout this solution, the symbol \( \star \) will stand for whatever notation your instructor prefers for using L’Hospital’s Rule on the indeterminate form 0/0; this may be “\( 0/0 \) or \( \frac{0}{0} \) or \( \frac{L}{L} \) or = “0/0” or “has the form \( \frac{0}{0} \), and so, by L’Hospital’s Rule, is equal to” or something else. The symbol \( \heartsuit \) will serve the same purpose for the indeterminate forms \( \infty/\infty \) and \( -\infty/\infty \).

(a) \( \lim_{x \to 1} \frac{x^3 - 1}{7 - 7x} \quad \star \quad \lim_{x \to 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7} \)

(b) \( \lim_{x \to 0} \frac{1 - \cos(2x)}{3x} = 0 \quad \frac{1}{1} = 0 \quad \text{Can’t use (and don’t need) L’Hospital’s Rule!} \)

(c) \( \lim_{x \to 0} \frac{1 - \cos(4x)}{5x^2} \quad \star \quad \lim_{x \to 0} \frac{4\sin(4x)}{10x} \quad \star \quad \lim_{x \to 0} \frac{16\cos(4x)}{10} = \frac{16}{10} = \frac{8}{5} \)

(d) \( \lim_{x \to \infty} \frac{x^2}{2^x} \quad \heartsuit \quad \lim_{x \to \infty} \frac{2x}{\ln 2 \cdot 2^x} \quad \heartsuit \quad \lim_{x \to \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0 \)

4. Consider the function \( f(x) = x^4e^x \) with domain all real numbers.

(a) Find the \( x \)-value(s) of all roots (\( x \)-intercepts) of \( f \).

The equation \( x^4e^x = 0 \) means \( x^4 = 0 \) (that is, \( x = 0 \)) or \( e^x = 0 \) (no solution), so the only root is at \( x = 0 \).

(b) Find the \( x \)- and \( y \)-value(s) of all critical points and identify each as a local max, local min, or neither.

\[
f'(x) = 4x^3e^x + x^4e^x
\]

\[
0 = x^3e^x(4 + x)
\]

\( \Rightarrow x = 0, -4 \quad \text{Note that } e^x \text{ is never } 0. \)

\[
\begin{array}{c|ccc}
\quad & x < -4 & -4 < x < 0 & 4 < x \\
\hline
f' & \text{positive} & \text{negative} & \text{positive} \\
\end{array}
\]

\( y \)-values: \( f(-4) = 256e^{-4} \approx 4.689, f(0) = 0 \)

So, \( f \) has a local maximum at \((-4, 256e^{-4})\) and a local minimum at \((0, 0)\).

(c) Find the \( x \)- and \( y \)-value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at \((0, 0)\). There is no global maximum because as \( x \to \infty, f(x) \to \infty \).

Note that as \( x \to -\infty, f(x) \to 0 \). You can verify this by using L’Hospital’s Rule on \( x^4/e^{-x} \).
(d) Find the x-value(s) of all inflection points.

\[ f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \]

Use Product Rule on each product in \( f'(x) \) above.

\[ 0 = e^x(x^4 + 8x^3 + 12x^2) \]
\[ 0 = e^x(x^2 + 8x + 12) \]
\[ 0 = e^x(x^2 + 2)(x + 6) \]
\[ \Rightarrow x = 0, -2, -6 \]

\[ \begin{array}{c|c|c|c|c}
  f'' & x < -6 & -6 < x < -2 & -2 < x < 0 & 0 < x \\
  f & \text{positive} & \text{negative} & \text{positive} & \text{positive} \\
\end{array} \]

So, the x-values of the inflection points of \( f \) are \( x = -2 \) and \( x = -6 \) but NOT \( x = 0 \).

(e) Sketch \( f \).

5. How would your answers to the previous question change if the domain of \( f \) were \([-10, 10]\)?

There would be a global maximum at \((10, 10^4e^{10})\). (And the graph would be restricted to \(-10 \leq x \leq 10\)).

6. Find an antiderivative of \( y = \frac{5}{\sqrt{1 - 9x^2}} + x^3 + \cos(2x) + e^3 \).

The answer is \( \frac{5 \arcsin 3x}{3} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^3x + C \), where \( C \) is any constant.

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is $288. If the glass for the sides costs $12 per square foot and the opaque material for the bottom costs $3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.

[Diagram of a box with dimensions labeled x, y, and z]

Goal: Maximize volume

Objective function: volume = \( V = x \cdot y \cdot z = x^2y \)
We need to get this down to a function of just one variable, so we use the constraint equation:

\[ \text{total cost} = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides}) \]

\[ 288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy \]

\[ 288 = 27xy + 24x^2 \]

\[ 288 - 24x^2 = 27xy \]

\[ \frac{288 - 24x^2}{27x} = y \]

Substituting this back into the objective function gives

\[ V = x^2y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27} (288x - 24x^3). \]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[ V'(x) = \frac{1}{27} (288 - 72x^2) \]

\[ 0 = \frac{1}{27} (288 - 72x^2) \]

\[ 0 = (288 - 72x^2) \]

\[ 72x^2 = 288 \]

\[ x^2 = \frac{288}{72} \]

\[ x = 2 \]

We discard \( x = -2 \) because lengths must be nonnegative.

Since \( V' \) is positive for \( x < 2 \) and negative for \( 2 < x \), we know that the maximum occurs at \( x = 2 \).

And

\[ y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9} \]

so the dimensions are 2 by 2 by \( \frac{32}{9} \).