1. Let \( f(x) = (\sqrt{x-1})^5 = (x-1)^{5/2} \).

1a. Use the following table of derivatives of \( f \) to find the third degree Taylor polynomial approximation \( P_3(x) \) of \( f(x) \), in powers of \( x-5 \), ie, \( x_0 = 5 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( f^{(k)}(x) )</th>
<th>( f^{(k)}(5) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>((x-1)^{5/2})</td>
<td>32</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{5}{2}(x-1)^{3/2})</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{15}{4}\sqrt{x-1})</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{15}{8}\sqrt{x-1})</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{15}{16}(x-1)^{-3/2})</td>
<td>-15</td>
</tr>
</tbody>
</table>

We need coeffs:

\[
C_0 = 32, \quad C_1 = 20, \quad C_2 = \frac{15/2}{2!} = \frac{15}{4}, \quad C_3 = \frac{15/16}{3!} = \frac{5}{32}
\]

\[
P_3(x) = 32 + 20(x-5) + \frac{15}{4}(x-5)^2 + \frac{5}{32}(x-5)^3
\]

1b. Taylor’s theorem guarantees that \( f \) and \( P_3 \) will separate by no more than what value (call it \( \epsilon \)), on the interval \([2.5, 6]\)? Show all your work. (Find \( K_4 \) to two decimal places. Find the maximum guaranteed error \( \epsilon \) to 4 decimal places.)

\[
|f(x) - P_3(x)| \leq \epsilon = \frac{K_4 |x-5|^4}{4!}
\]

where \(|f^{(4)}(x)| \leq K_4 \) on \([2.5, 6]\).

Graphical analysis and tables imply the smallest \( K_4 \) to two decimal places is 0.52

Also, the max of |x-5|^4 occurs at x = 2.5

\[
\epsilon = \frac{0.52 \times 2.5^4}{4!} \approx 0.8464
\]

1c. Since \( f(x) = (\sqrt{x-1})^5 \), we could use it to find \((\sqrt{3})^5 = 3^{5/2}\) by taking \( x = 4 \). Use the same \( x \) to approximate \((\sqrt{3})^5\) with your Taylor polynomial \( P_3(x) \). What do you get? What’s the error in this approximation? (Write your answers to 4 decimal places).

By calculator, \(3^{5/2} \approx 15.5885\)

And \( P_3(4) = 15.5938 \); the difference is \( \approx 0.0053\).
2. Find \( \int \frac{x^3 - 4x}{\sqrt{4 + x^2}} \, dx \) by introducing an appropriate trig substitution, say, \( x = 2 \tan t \). Express your answer in terms of \( x \) and \( \sqrt{4 + x^2} \) (leave no trig functions in your answer if possible).

Let \( x = 2 \tan t \). Then \( x^3 = 8 \tan^3 t \), \( 4x = 8 \tan t \), \( dx = 2 \sec^2 t \, dt \).

and \( \sqrt{4 + x^2} = \sqrt{4 + 4 \tan^2 t} = 2 \sqrt{1 + \tan^2 t} = 2 \sec t \).

Making all these substitutions, we get \( \int \frac{x^3 - 4x}{\sqrt{4 + x^2}} \, dx = \int \frac{8 \tan^3 t - 8 \tan t}{2 \sec t} \, 2 \sec^2 t \, dt \).

\( = 8 \int (\tan^2 t - 1) \sec t \, dt \).

Next, since \( \tan^2 t = (\sec^2 t - 1) \) we rewrite this now as

\( = 8 \int (\sec^2 t - 2) \sec t \, dt \).

\( = 8 \int (u^2 - 2) \, du \).

Now, from the very 1st line, \( x = 2 \tan t \Rightarrow \frac{x}{2} = \tan t = \frac{opp}{adj} \) in this right triangle.

Now, \( \sec t = \frac{hyp}{adj} = \frac{\sqrt{4 + x^2}}{2} \) so \( \int \sec t \, dt = \ln \left( \frac{\sqrt{4 + x^2} + 2}{2} \right) + C \) becomes

\( \frac{8}{3} \left( \frac{\sqrt{4 + x^2}}{2} \right)^3 - 16 \frac{\sqrt{4 + x^2}}{2} + C \).

Or, \( \frac{1}{3} (4 + x^2)^{3/2} - 8 \sqrt{4 + x^2} + C \).

3. At some point, most calculus students learn that \( \frac{d}{dt} \arcsin t = \frac{1}{\sqrt{1 - t^2}} \). But to find \( \int \arcsin x \, dx \) requires integration by parts. Show how.

"LIALTE" suggests letting \( u \) be the Inverse trig function, so let \( u = \arcsin x \).

WOW! we've been GIVEN (reminded of) \( u' \) in the first sentence!!! (\( \frac{1}{\sqrt{1 - x^2}} \)

So:

\( \begin{align*}
  u &= \arcsin x \\
  u' &= \frac{1}{\sqrt{1 - x^2}} \\
  v &= x \\
  v' &= dx
\end{align*} \)

\( \Rightarrow \int \arcsin x \, dx = uv - \int v'u \, dx \).

This last integral can be done quickly by a substitution:

Let \( u = 1 - x^2 \); \( dw = -2x \, dx \) and so \( -\frac{x}{\sqrt{1 - x^2}} \, dx = -\frac{1}{2} \int \frac{-2x \, dx}{\sqrt{1 - x^2}} = \frac{1}{2} \int \frac{dw}{\sqrt{w}} \).

So:

\( w = 1 - x^2 \); \( dw = -2x \, dx \) and so \( -\frac{x}{\sqrt{1 - x^2}} \, dx \) becomes \( \frac{1}{2} \int \frac{dw}{\sqrt{w}} = \frac{1}{2} \left( \frac{w^{1/2}}{1/2} \right) + C = \sqrt{w} + C = \sqrt{1 - x^2} + C \).

Finally:

\( \int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + C \).
4A. Solve the differential equation \( \frac{dy}{dx} = \frac{x^5}{y} \). Write your answer as \( y = \ldots \)

\[
\frac{dy}{dx} = \frac{x^5}{y} \quad \Leftrightarrow \quad y \frac{dy}{dx} = x^5 dx \\
\Leftrightarrow \int y \, dy = \int x^5 \, dx \\
\Leftrightarrow \frac{y^2}{2} = \frac{x^6}{6} + C_1 \\
\Leftrightarrow y^2 = \frac{x^6}{3} + 2C_1 = \frac{x^6}{3} + C \\
\Leftrightarrow y = \pm \sqrt{\frac{x^6}{3} + C}
\]

4B. Your answer to 4A is a whole family of solutions determined by choice of some constant. Which one of those solutions is the solution to the initial value problem: \( \frac{dy}{dx} = \frac{x^5}{y} \); \( y(3) = 16 \)? Again, write your answer as \( y = \ldots \)

We need to choose the solution \( y(y) \) which makes \( y(3) = 16 \), that is, \( y = 16 \) for \( x = 3 \).

Since 16 is positive, we need \( y = +\sqrt{\frac{x^6}{3} + C} \) \([\text{if given } y(3) = -16, \text{we'd take}]

\[ y = -\sqrt{\frac{x^6}{3} + C} \]

Now, \( y = 16 \) and \( x = 3 \) mean:

\[
16 = \sqrt{\frac{3^6}{3} + C} \\
= \sqrt{3^5 + C} \\
= \sqrt{243 + C}
\]

So \( 16^2 = 243 + C \)

\( 256 = 243 + C \)

Finally:

\( y = \sqrt{\frac{x^6}{3} + 13} \)

NOTE WELL!

You cannot replace

\( y = \pm \sqrt{\frac{x^6}{3} + C} \) with \( \sqrt{x^6} + K \).

(The \( C \) doesn't "come out from under the \( \sqrt{} \)."

If this was OK, then we'd have

\[ 16 = \sqrt{\frac{3^6}{3} + K} \]

\[ 16 = \sqrt{243} + K \]

\[ K = 16 - \sqrt{243} = 0.41154... \]

But if you graph \( \sqrt{\frac{x^6}{3} + 13} \) and \( \sqrt{\frac{x^6}{3} + 0.4115} \) together, you'll see they are DIFFERENT functions!

(in particular, they have intercepts of \( \sqrt{3} \approx 3.0 \) and \( 0.4115 \) respectively!)
5A. Find \( \int \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx \).
Show all your work, from setting up the partial fraction decomposition to solving for \( A, B, C \), etc, to final integrations.

\[
\int \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx = \int \frac{A}{1 + x^2} + \frac{B}{x - 3} + \frac{C}{x + 3} \, dx
\]

Note: \( 4x - 5 + 7x^2 = (A + B)(x - 3) + C(1 + x^2) \)

If \( x = 3 \):
\[
70 = (A \cdot 3 + B)0 + C(1 + 9) \Rightarrow 70 = 10C \Rightarrow [C = 7]
\]

If \( x = 0 \):
\[
-5 = (A \cdot 0 + B)(-3) + 7(1 + 0^2) \Rightarrow -5 = 3B + 7 \Rightarrow 12 = 3B \Rightarrow [B = 4]
\]

If \( x = 1 \):
\[
12 = (A \cdot 1 + 4)(1) + 7(1 + 6) \Rightarrow 12 = 4A + 4 + 19 \Rightarrow [A = 0]
\]

(Or, if \( x = 1 \):
\[
6 = (A \cdot 1 + 4)(-2) + 7(1 + 1) \Rightarrow 6 = -2A - 8 + 14 \Rightarrow 6 = -2A + 6 \Rightarrow [A = 0]
\]

So
\[
\int \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx = \int \frac{0x + 4}{1 + x^2} + \frac{7}{x - 3} \, dx
\]

\[
= 4 \int \frac{dx}{1 + x^2} + 7 \int \frac{dx}{x - 3}
\]

\[
= 4 \arctan x + 7 \ln |x - 3| + K
\]

(best not to write "C" here since C has already been used in this problem)

5B. Explain why \( \int_0^3 \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx \) is an improper integral.

The integrand, \( \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \), has a vertical asymptote at \( x = 3 \):

5C. Does the integral in 5B converge? Explain! (You do not have to make a table. Just make a good argument, knowing standard behavior of the functions involved in the anti-derivative in 5A).

By definition, \( \int_0^3 \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx = \lim_{B \to 3^-} \int_0^B \frac{4x - 5 + 7x^2}{(1 + x^2)(x - 3)} \, dx = \lim_{B \to 3^-} \left[ 4 \arctan x + 7 \ln |x - 3| \right]_0^B \)

Now, as \( B \to 3^- \), \( x - 3 \to 0^- \), so \( 7 \ln |x - 3| \to -\infty \); hence the integral in 5B DIVERGES.

Note: the "p-test" is not helpful here:

It's for integrals of the form \( \int_1^\infty \) ...
6. Here is the graph of \( f(x) = \begin{cases} 
0 & \text{if } x < -1 \\
0.25(x + 1) & \text{if } -1 \leq x < 1 \\
\lambda e^{-\lambda x} & \text{if } 1 \leq x 
\end{cases} \)

for a certain value of \( \lambda \):

6a. By definition, what is the total area of the entire (both the light and dark parts combined) shaded region required to be in order for \( f \) to be a probability density function?

\[ \text{Area must be 1} \]

6b. It's easy to find the area of the lightly-shaded region. What is it?

\[ \text{it's the area of a triangle} = \frac{1}{2} \cdot \text{base} \cdot \text{height} = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} = \frac{1}{2} \]

6c. Find \( \lambda \) so that the area of the darkly-shaded region is what it needs to be, according to your answers in 6a and 6b. You may use without verification this fact: \( \int_{a}^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda a} \).

The area of the dark region must also be \( \frac{1}{2} \) so that the TOTAL area is 1.

The area of the dark region is \( \frac{1}{2} = \int_{a=1}^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda} \)

so \( \frac{1}{2} = e^{-\lambda} \) \( \ln \frac{1}{2} = -\lambda \); \( \lambda = -\ln \frac{1}{2} = (-\ln(0.5)) \)

so \( \lambda = 0.6931 \ldots \)

6d. Suppose that \( X \) is a random variable with this \( f \) as its probability density function. What's the probability that \( X \) is negative?

It's the area of the little \( \triangle \) marked \( \smiley \), which is \( \frac{1}{2} \cdot 1 \cdot 0.25 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8} \)

6e. What's the probability that \( X \) is in the interval \([0, 1]\)?

It's the area of the trapezoid next to the \( \bigcirc \) marked \( T' \) which is \( \frac{1}{2} \left( \frac{1}{2} + 1 \right) \cdot \frac{3}{4} = \frac{3}{8} \)

6f. What's the probability that \( X \) is greater than 4?

\[ \int_{4}^{\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda} \cdot \frac{3}{4} \]

\[ = e^{-\lambda} \cdot \frac{3}{4} = \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8} \]