1. (a) \( \int \sec^3 x \tan^3 x \, dx \)

Start by splitting off one factor of \( \sec x \tan x \), use a trigonometric identity to write the tangent function in terms of secant, then substitute \( u = \sec x \).

\[
\int \sec^3 x \tan^3 x \, dx = \int \sec^2 x \tan x \, dx
\]
\[
= \int (\sec^4 x - \sec^2 x) \sec x \, dx \quad \text{(since \( \tan^2 x = \sec^2 x - 1 \))}
\]
\[
= \int (u^4 - u^2) \, du \quad \text{(where \( u = \sec x \) and \( du = \sec x \tan x \, dx \))}
\]
\[
= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C
\]

(b) \( \int \theta \sec \theta \tan \theta \, d\theta \)

To start, apply integration by parts with

\[
u = \sec \theta \tan \theta \quad \Rightarrow \quad d\theta = \sec \theta \tan \theta \, d\theta
\][
\[d\theta = \sec \theta \tan \theta \, d\theta \quad \Rightarrow \quad v = \sec \theta
\]

\[
\int \theta \sec \theta \tan \theta \, d\theta = \theta \sec \theta - \int \sec \theta \, d\theta = \theta \sec \theta - \ln |\sec \theta + \tan \theta| + C
\]

(c) \( \int \frac{3x + 4}{(x - 2)(x^2 + 1)} \, dx \)

Use partial fractions:

\[
\int \frac{3x + 4}{(x - 2)(x^2 + 1)} \, dx = \int \left( \frac{A}{x - 2} + \frac{Bx + D}{x^2 + 1} \right) \, dx
\]

This gives \( 3x + 4 = A(x^2 + 1) + (Bx + D)(x - 2) \) therefore

\[
\begin{cases}
  x = 2 & \Rightarrow A = 2 \\
  x = 0 & \Rightarrow D = -1 \\
  x = 1 & \Rightarrow B = -2
\end{cases}
\]

\[
\int \frac{3x + 4}{(x - 2)(x^2 + 1)} \, dx = \int \left( \frac{2}{x - 2} - \frac{2x - 1}{x^2 + 1} \right) \, dx
\]
\[
= \int \frac{2}{x - 2} \, dx - \int \frac{2x}{x^2 + 1} \, dx - \int \frac{1}{x^2 + 1} \, dx
\]
\[
\text{(let} \ w = x - 2 \text{and} \ dw = dx; \text{& let} \ u = x^2 + 1 \text{and} \ du = 2x \, dx)
\]
\[
= 2 \left[ \int \frac{1}{w} \, dw - \int \frac{1}{u} \, du - \int \frac{1}{x^2 + 1} \, dx \right]
\]
\[
= 2 \ln |w| - \ln |u| - \arctan x + C
\]
\[
= 2 \ln |x - 2| - \ln |x^2 + 1| - \arctan x + C
\]
2. The ellipse to the right is given by \( x^2 + 4y^2 = 1 \).

Solve for \( y \) to find the function that represents the top half of the ellipse:

\[ x^2 + 4y^2 = 1 \iff 4y^2 = 1 - x^2 \iff y = \frac{1}{2}\sqrt{1 - x^2}. \]

An integral which represents the top half of the ellipse is given by

\[ \frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} \, dx \]

Use trigonometric substitution with \( x = \sin \theta \) for \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\), then \( dx = \cos \theta \, d\theta \).

Use \( \sin \theta = x \) to construct a triangle with the opposite side of length \( x \) and hypotenuse of length \( 1 \).

The triangle gives:

\[ \cos \theta = \sqrt{1 - x^2}. \]

Likewise, our substitution gives \( \theta = \arcsin x \).

To change limits of integration:

\[ x = -1 \Rightarrow \theta = \arcsin(-1) = -\frac{\pi}{2} \]
\[ x = 1 \Rightarrow \theta = \arcsin(1) = \frac{\pi}{2} \]

\[
\frac{1}{2} \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2 \theta} \cos \theta \, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sqrt{\cos^2 \theta} \cos \theta \, d\theta \quad \text{(since } \cos^2 \theta = 1 - \sin^2 \theta) \\
= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = \frac{1}{4} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) \, d\theta \quad \text{(since } \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)) \\
= \frac{1}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \frac{1}{4} \left[ \left( \frac{\pi}{2} + \frac{1}{2} \sin \pi \right) - \left( -\frac{\pi}{2} + \frac{1}{2} \sin(-\pi) \right) \right] \\
= \frac{1}{4} \left( \frac{\pi}{2} + \frac{\pi}{2} \right) = \frac{\pi}{4}
\]

3. Evaluate \( \int_{3}^{6} \frac{1}{(4-x)^2} \, dx \).

\[
\int_{3}^{6} \frac{1}{(4-x)^2} \, dx = \int_{3}^{4} \frac{1}{(4-x)^2} \, dx + \int_{4}^{6} \frac{1}{(4-x)^2} \, dx \quad \text{(since } \frac{1}{(4-x)^2} \text{ is undefined at } x = 4) \\
= \lim_{b \to 4^-} \int_{3}^{b} \frac{1}{(4-x)^2} \, dx + \lim_{a \to 4^+} \int_{a}^{6} \frac{1}{(4-x)^2} \, dx \quad \text{(since each integral is improper)} \\
\text{(let } u = 4 - x \text{ then } du = -dx) \\
= \lim_{b \to 4^-} \int_{1}^{4-b} (-u^{-2}) \, du + \lim_{a \to 4^+} \int_{4-a}^{2} (-u^{-2}) \, du = \lim_{b \to 4^-} \left[ \frac{1}{u} \right]_{1}^{4-b} + \lim_{a \to 4^+} \left[ \frac{1}{u} \right]_{4-a}^{-2} \\
= \lim_{b \to 4^-} \left( \frac{1}{4 - b} - 1 \right) + \lim_{a \to 4^+} \left( -\frac{1}{2} - \frac{1}{4 - a} \right) \\
\text{But } \lim_{b \to 4^-} \left( \frac{1}{4 - b} - 1 \right) = \infty \text{ and } \lim_{a \to 4^+} \left( -\frac{1}{2} - \frac{1}{4 - a} \right) = \infty.
\]

Therefore, \( \int_{3}^{6} \frac{1}{(4-x)^2} \, dx \) diverges.
4. Consider the function \( f(x) = e^{-2x} \).

(a) The 3rd-order Taylor polynomial, \( P_3(x) \), of \( f(x) \) centered at \( x = 0 \) may be found as follows:

\[
\begin{align*}
  f(x) &= e^{-2x} \\
  f'(x) &= -2e^{-2x} \\
  f''(x) &= 4e^{-2x} \\
  f'''(x) &= -8e^{-2x}
\end{align*}
\]

\[
P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3
\]

Therefore, \( P_3(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 \).

(b) Use \( P_3(x) \) to find an estimate for \( \frac{1}{e} \).

Note: \( f\left(\frac{1}{2}\right) = e^{-1} = \frac{1}{e} \), so we can estimate this value by calculating \( P_3\left(\frac{1}{2}\right) \), since \( P_3(x) \approx f(x) \) for values of \( x \) near 0. Therefore, \( P_3\left(\frac{1}{2}\right) = 1 - 2 \left(\frac{1}{2}\right) + 2 \left(\frac{1}{2}\right)^2 - \frac{4}{3} \left(\frac{1}{2}\right)^3 = \frac{1}{3} \) approximates \( \frac{1}{e} \).

(c) We can use Taylor’s Theorem to approximate the error of our estimate from part (b) on the interval \([0, \frac{1}{2}]\). Recall that error bounds for estimates using the 3rd-order Taylor Polynomial may be determined using:

\[
|f(x) - P_3(x)| \leq \frac{K_4}{4!}|x-0|^4.
\]

To find \( K_4 \), notice that since \( f^{(4)}(x) = 16e^{-2x} \) is decreasing it attains its maximum when \( x = 0 \). So, \( |f^{(4)}(x)| \leq |16e^{-2(0)}| = 16 = K_4 \) on \([0, \frac{1}{2}]\). Likewise, \(|x-0|^4 \) is greatest on \([0, \frac{1}{2}]\) when \( x = \frac{1}{2} \).

Therefore, \( |f(x) - P_3(x)| \leq \frac{16}{4!} \left|\frac{1}{2} - 0\right|^4 = \frac{1}{24} \).

5. (a) Recall from class: \( \int_1^\infty \frac{1}{x^p} \, dx \) converges for \( p > 1 \) and diverges otherwise. Thus, we know the area under \( 1/x \) for \( x > 1 \) is infinite since \( \int_1^\infty \frac{1}{x} \, dx \) diverges. Likewise, the area under \( 1/x^2 \) for \( x > 1 \) is finite since \( \int_1^\infty \frac{1}{x^2} \, dx \) converges. You may think of finding the area between the two functions, for \( x > 1 \), as the result of removing a finite area from an infinitely large area, therefore, the result must be infinite as well. (Or, if you prefer a more rigorous solution, notice that the shaded area between \( y = \frac{1}{x} \) and \( y = \frac{1}{x^2} \) for \( x > 1 \) is given by the integral \( \int_1^b \left( \frac{1}{x} - \frac{1}{x^2} \right) \, dx \) which diverges since \( \lim_{b \to \infty} \int_1^b \left( \frac{1}{x} - \frac{1}{x^2} \right) \, dx = \lim_{b \to \infty} \left[ \ln x + \frac{1}{x} \right]_1^b = \lim_{b \to \infty} \left[ \ln b + \frac{1}{b} \right] - (0 + 1) = \infty \).

(b) i. Note: \( \int_1^\infty \frac{1}{x^2} \, dx \) converges and \( 0 \leq f(x) \leq \frac{1}{x^2} \) for all \( x \geq 1 \), therefore \( \int_1^\infty f(x) \, dx \) converges by comparison.

ii. Note: \( \int_1^\infty \frac{1}{x} \, dx \) diverges and \( 0 \leq \frac{1}{x} \leq g(x) \) for values of \( x \) greater than approximately 2.2 (see graph), therefore \( \int_1^\infty g(x) \, dx \) diverges by comparison.

iii. Note: \( \frac{1}{x^2} \leq h(x) \leq \frac{1}{x} \), therefore it is impossible to tell if \( \int_1^\infty h(x) \, dx \) converges given the information provided.
6. A 12-inch bar that is clamped at both ends is to be subjected to an increasing amount of stress until it snaps. Let $X$ be the distance from the left end at which the break occurs. Suppose that $X$ has a probability density function given by:

$$f(x) = \begin{cases} 
  \frac{x}{24} \left(1 - \frac{x}{12}\right) & \text{if } 0 \leq x \leq 12 \\
  0 & \text{otherwise}
\end{cases}$$

(a) The probability that the break point occurs between 2 and 4 inches from the left end of the bar is given by:

$$\int_2^4 \frac{x}{24} \left(1 - \frac{x}{12}\right) \, dx = \frac{1}{24} \int_2^4 \left(x - \frac{x^2}{12}\right) \, dx = \frac{1}{24} \left[ \frac{x^2}{2} - \frac{x^3}{36} \right]_2^4 = \frac{5}{27}$$

(b) The expected break point of the bar is given by:

$$\int_{-\infty}^{\infty} x f(x) \, dx = \int_{-\infty}^{0} x f(x) \, dx + \int_{0}^{12} x f(x) \, dx + \int_{12}^{\infty} x f(x) \, dx$$

$$= \int_{-\infty}^{0} 0 \, dx + \int_{0}^{12} x \left(\frac{x}{24} - \frac{x^2}{288}\right) \, dx + \int_{12}^{\infty} 0 \, dx \quad \text{(refer to definition of } f \text{ above)}$$

$$= \frac{1}{24} \int_{0}^{12} \left(x^2 - \frac{x^3}{12}\right) \, dx$$

$$= \frac{1}{24} \left[ \frac{x^3}{3} - \frac{x^4}{48} \right]_0^{12} = 6$$

The expected break point of the bar is 6 inches from the left end.

7. Consider $\int_4^{\infty} \frac{3 + \sin \theta}{\theta^3} \, d\theta$.

We have: $-1 \leq \sin \theta \leq 1$ for all values of $\theta$

$$2 \leq 3 + \sin \theta \leq 4 \quad \text{for all values of } \theta$$

Thus, $0 \leq \frac{3 + \sin \theta}{\theta^3} \leq \frac{4}{\theta^3}$ for $\theta > 0$

Consider the comparison integral,

$$\int_4^{\infty} \frac{4}{\theta^3} \, d\theta = \lim_{b \to \infty} \int_4^{b} \frac{4}{\theta^3} \, d\theta = \lim_{b \to \infty} \left[ -\frac{2}{\theta^2} \right]_4^b$$

$$= \lim_{b \to \infty} \left[ -\frac{2}{b^2} + \frac{2}{4^2} \right] = \frac{1}{8}.$$

Therefore, $\int_4^{\infty} \frac{3 + \sin \theta}{\theta^3} \, d\theta$ converges by comparison. In fact, $\int_4^{\infty} \frac{3 + \sin \theta}{\theta^3} \, d\theta \leq \frac{1}{8}$.