INTEGRATION TIPS

• Substitution: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

• Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial”.)

- Logarithms (such as \( \ln x \))
- Inverse trig (such as \( \arctan x, \arcsin x \))
- Algebraic (such as \( x, x^2, x^3 + 4 \))
- Trig (such as \( \sin x, \cos 2x \))
- Exponentials (such as \( e^x, e^{3x} \))

• Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E\) here) above it on the right.

• Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Substitution</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( x = a \sin t )</td>
<td>( \sin^2 x = \frac{1}{2} - \cos^2(2x) )</td>
</tr>
<tr>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( x = a \tan t )</td>
<td>( \tan^2 x + 1 = \sec^2 x )</td>
</tr>
<tr>
<td>( \sqrt{x^2 - a^2} )</td>
<td>( x = a \sec t )</td>
<td>( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} )</td>
</tr>
</tbody>
</table>

| \( \sin(2x) \) | \( = 2 \sin x \cos x \) |

• Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m, n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use integration by parts or identities shown to right of this box.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} ) ( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} )</td>
</tr>
</tbody>
</table>
\[ \int \tan^m x \sec^n x \, dx \]

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<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sec x \tan x ) and substitute ( u = \sec x ).</td>
</tr>
<tr>
<td>( n ) even</td>
<td>Break off one factor of ( \sec^2 x ) and substitute ( u = \tan x ).</td>
</tr>
<tr>
<td>( m ) even, ( n ) odd</td>
<td>Use identity at right to reduce to powers of ( \sec x ) alone. Then use integration by parts or reduction formula (if allowed).</td>
</tr>
</tbody>
</table>

**Useful Trigonometric Derivatives and Antiderivatives**

\[
\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln|\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\lim_{x \to \infty} e^x = \infty \quad \lim_{x \to \infty} e^{-x} = 0 \quad \lim_{x \to \infty} \frac{1}{x} = 0 \quad \lim_{x \to \infty} \ln x = \infty \quad \lim_{x \to 0^+} \ln x = -\infty \quad \lim_{x \to \infty} \arctan x = \frac{\pi}{2}
\]

1. Evaluate the following.

(a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx
\]

\[
= \int u^6 (1 - u^2) \, du
\]

\[
= \int (u^6 - u^8) \, du
\]

\[
= \frac{u^7}{7} - \frac{u^9}{9} + C
\]

\[
= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]

(b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

\[
y^2 + 100 = y^2 \Rightarrow y = \sqrt{x^2 + 100}
\]

\[
\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10}
\]

\[
\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}} \\
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}} \\
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}} \\
= \int \sec t \, dt \\
= \ln |\sec t + \tan t| + C \\
\]

Now use \(1 + \tan^2 t = \sec^2 t\).

\[
= \ln |\sqrt{x^2 + 100} + \frac{x}{10}| + C \\
\]

Now use triangle above.

(c) This is an improper integral, so we need to use a limit.

\[
\int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx = \lim_{t \to \infty} \int_3^t \frac{1}{x(\ln x)^{100}} \, dx \\
= \lim_{t \to \infty} \int_{x=3}^{x=t} \frac{1}{u^{100}} \, du \\
= \lim_{t \to \infty} \left[ \frac{u^{-99}}{-99} \right]_{x=3}^{x=t} \\
= \lim_{t \to \infty} \left[ \frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \\
= 0 - \frac{-1}{99(\ln 3)^{99}} \\
= \frac{1}{99(\ln 3)^{99}} \\
\]

So, the integral converges (to this value).

(d) We’ll use integration by parts: \(u = x \Rightarrow du = dx\) and \(dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}\).

\[
\int_0^\infty xe^{-2x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ xe^{-2x} \right]_0^t - \int_0^t e^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ xe^{-2x} \right]_0^t - \frac{e^{-2x}}{4} \bigg|_0^t \\
= \lim_{t \to \infty} \left[ -\frac{x}{2e^{2x}} - \frac{1}{4e^{2x}} \right]_0^t \\
= \lim_{t \to \infty} \left[ -\frac{t}{2e^{2t}} - \frac{1}{4e^{2t}} \right] - \left[ 0 - \frac{1}{4e^0} \right] \\
= (0 - 0) - (0 - 1/4) \\
= 1/4 \\
\]

So, the integral converges (to this value).

(e) Partial Fractions:

Write \(\frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3}\). Now multiply both sides by \((x - 3)(x^2 + 1)\) to get
\[ 3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1). \]

Let \( x = 3 \). Then \( 20 = C(10) \), so \( C = 2 \).

Let \( x = 0 \). Then \( -13 = B(-3) + 2(1) \), so \( B = 5 \).

Let \( x = 1 \). Then \( -8 = (A(1) + 5)(-2) + 2(2) \), so \( A = 1 \).

\[ \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx \]

Let \( u = x^2 + 1 \), so \( du = 2xdx \).

\[ = \frac{\ln u}{2} + 5 \arctan x + 2 \ln |x - 3| + D \]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
\begin{array}{r|rrrr}
& 4x^2 & -3x & 2 & -5 \\
\hline
x - 6 & 4x^3 & -27x^2 & +20x & -17 \\
& 4x^3 & -24x^2 & & \\
& -3x^2 & & & \\
& -3x^2 & +18x & & \\
& 2x & & & \\
& 2x & -12 & & \\
& -5 & & & \\
\end{array}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \( x = 1 \) because the integrand has a vertical asymptote there.

\[
\int_1^3 \frac{1}{x-1} \, dx = \lim_{t \to 1^+} \int_t^3 \frac{1}{x - 1} \, dx = \lim_{t \to 1^+} \ln |x - 1|_t^3 = \lim_{t \to 1^+} [\ln |3 - 1| - \ln |t - 1|]
\]

Since \( \lim_{t \to 1^+} (-\ln |t - 1|) = \infty \), this integral diverges (to \( \infty \)).

2. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

\[
f(x) = x^{1/2} \]
\[
f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2 \sqrt{x}} \]
\[
f''(x) = -\frac{1}{4} x^{-3/2} = -\frac{1}{4x^{3/2}} \]
\[
f(100) = 10 \]
\[
f'(100) = \frac{1}{2 \sqrt{100}} = \frac{1}{20} \]
\[
f''(100) = -\frac{1}{4 \cdot 100^{3/2}} = -\frac{1}{4000} \]
\[ P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2 \]
\[ = 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000} \]

3. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval \([100, 110]\)?

We know that \(|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}.

In this case, \(n = 2\), \(x_0 = 100\), and \(x = 110\) (the farthest from \(x_0\) that we are considering).

\[ K_3 = \text{max of } |f'''(x)| \text{ on } [100, 110] = \text{max of } \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 110] = \frac{3}{8 \cdot 100^{3/2}} = \frac{3}{800,000} \]

Putting this all together, we have \(|f(x) - P_2(x)| \leq \frac{3}{3!}110 - 100)^3 = \frac{1}{1600}.

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

For all \(x \geq 1\), we have \(\frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \(\cos x\) is \(-1\).

Since \( \int_1^\infty \frac{5}{x^{0.99}} \, dx \) diverges (compute yourself or notice that \(p = 0.99 < 1\)), we know that the integral in question must diverge too.

(b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

For all \(x \geq 1\), we have \(\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2} \). (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[ 4 \int_1^\infty \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = 4 \lim_{t \to \infty} -\frac{1}{x} \bigg|_1^t = 4 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{1}{1} \right] = 4[0 - (-1)] = 4 \]

Therefore, the original integral in question must converge to a value less than 4.