INTEGRATION TIPS

- Substitution: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPE will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial”.)

- Logarithms (such as \( \ln x \))
- Inverse trig (such as \( \arctan x, \arcsin x \))
- Algebraic (such as \( x, x^2, x^3 + 4 \))
- Trig (such as \( \sin x, \cos 2x \))
- Exponentials (such as \( e^x, e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

- Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x+1)(x-3)(x^2+5)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \( (x - 3) \) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \( (Dx + E) \) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>( x = a \sin t )</th>
<th>( x = a \tan t )</th>
<th>( x = a \sec t )</th>
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<tr>
<td>( \sqrt{a^2 - x^2} )</td>
<td>( \sqrt{a^2 + x^2} )</td>
<td>( \sqrt{x^2 - a^2} )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\sin^2 x + \cos^2 x = 1 \\
\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 = \sec^2 x \\
\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) = 2 \sin x \cos x
\]

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
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<tr>
<td>( m ) odd | Break off one factor of ( \sin x ) and substitute ( u = \cos x ). | ( \sin^2 x = 1 - \cos^2 x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n ) odd | Break off one factor of ( \cos x ) and substitute ( u = \sin x ). | ( \cos^2 x = 1 - \sin^2 x )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( m, n ) even | Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use integration by parts or identities shown to right of this box. | ( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} ) | ( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} )</td>
<td></td>
<td></td>
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</table>
\[ \int \tan^m x \sec^n x \, dx \]

| m even, n odd | Use identity at right to reduce to powers of \( \sec x \) alone. | \( \tan^2 x = \sec^2 x - 1 \) |
| m odd | Break off one factor of \( \sec x \tan x \) and substitute \( u = \sec x \). | \( \tan^2 x = \sec^2 x - 1 \) |
| n even | Break off one factor of \( \sec^2 x \) and substitute \( u = \tan x \). | \( \sec^2 x = \sec^2 x + 1 \) |

Possible Strategy

Identity to Use

| \( \frac{d}{dx} \tan x = \sec^2 x \) | \( \frac{d}{dx} \sec x = \sec x \tan x \) | \( \int \sec x \, dx = \ln |\sec x + \tan x| + C \) |

Useful Trigonometric Derivatives and Antiderivatives

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

  \[ \lim_{x \to \infty} e^x = \infty \]
  \[ \lim_{x \to \infty} e^{-x} = 0 \]
  \[ \lim_{x \to \infty} \frac{1}{x} = 0 \]
  \[ \lim_{x \to 0^+} \ln x = \infty \]
  \[ \lim_{x \to 0^+} \ln x = \infty \]
  \[ \lim_{x \to \infty} \arctan x = \pi/2 \]

1. Evaluate the following.

   (a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

   \[
   \int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \\
   = \int u^6 (1 - u^2) \, du \\
   = \int (u^6 - u^8) \, du \\
   = \frac{u^7}{7} - \frac{u^9}{9} + C \\
   = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
   \]

   Use \( \cos^2 x = 1 - \sin^2 x \).

   (b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

   \[
   x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
   \sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
   \tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
   \]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}}
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}}
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}}
= \int \sec t \, dt
= \ln|\sec t + \tan t| + C
\]
Now use triangle above.

(c) This is an improper integral, so we need to use a limit.
\[
\int_{3}^{\infty} \frac{1}{x \ln x} \frac{dx}{100} = \lim_{t \to \infty} \int_{3}^{t} \frac{1}{x \ln x} \frac{dx}{100}
= \lim_{t \to \infty} \int_{3}^{u} \frac{1}{u^{100}} \frac{du}{x}
= \lim_{t \to \infty} \frac{u^{-99}}{-99} \bigg|_{x=3}^{x=t}
= \lim_{t \to \infty} \left[ \frac{-1}{99(\ln x)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right]
= 0 - \frac{-1}{99(\ln 3)^{99}}
= \frac{1}{99(\ln 3)^{99}}
\]
Now use triangle above.

(d) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2} \).
\[
\int_{0}^{\infty} xe^{-2x} \, dx = \lim_{t \to \infty} \int_{0}^{t} xe^{-2x} \, dx
= \lim_{t \to \infty} \left[ \frac{-xe^{-2x} + e^{-2x}}{-2} \right]_{0}^{t}
= \lim_{t \to \infty} \left[ e^{-2x} - \frac{1}{2e^{2x}} \right]_{0}^{t}
= \lim_{t \to \infty} \left[ \frac{e^{-2t} - \frac{1}{4e^{2t}}}{2e^{2t}} - \left( 0 - \frac{1}{4e^{0}} \right) \right]
= (0 - 0) - (0 - 1/4)
= 1/4
\]
So, the integral converges (to this value).

(e) Partial Fractions:
Write \( \frac{3x^2 + 2x - 13}{(x-3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x-3} \). Now multiply both sides by \( (x-3)(x^2 + 1) \) to get
\[ 3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1). \]

Let \( x = 3 \). Then \( 20 = C(10) \), so \( C = 2 \).

Let \( x = 0 \). Then \( -13 = B(-3) + 2(1) \), so \( B = 5 \).

Let \( x = 1 \). Then \( -8 = (A(1) + 5)(-2) + 2(2) \), so \( A = 1 \).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \( u = x^2 + 1 \), so \( du = 2xdx \).

\[
= \frac{1}{2} \ln |x^2 + 1| + 5 \arctan x + 2\ln |x - 3| + D
\]

\( f \) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
x - 6 \quad \frac{4x^3 - 27x^2 + 20x - 17}{4x^3 - 24x^2 + 3x^2 + 18x - 2x - 12}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5\ln |x - 6| + C
\]

\( g \) This integral is improper at \( x = 1 \) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{5} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to 1^+} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} \bigg|_{-1}^{5} + \lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} \bigg|_{b}^{5}
\]

\[
= \lim_{a \to 1^-} \left[ \frac{-1}{5(a - 1)^5} - \frac{-1}{5(-1 - 1)^5} \right] + \lim_{b \to 1^+} \left[ \frac{-1}{5(5 - 1)^5} - \frac{-1}{5(b - 1)^5} \right]
\]

Since \( \lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} = \infty \) and \( \lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} = \infty \), this integral diverges (to \( \infty \)).
2. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

\[
\begin{align*}
f(x) &= x^{1/2} \\
f'(x) &= \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\
f''(x) &= \frac{-1}{4} x^{-3/2} = \frac{-1}{4x^{3/2}}
\end{align*}
\]

\[
f(100) = 10 \\
f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20} \\
f''(100) = \frac{-1}{4 \cdot 100^{3/2}} = \frac{-1}{4000}
\]

\[
P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2
\]

\[
= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
\]

3. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval \([100, 110]\)?

We know that  
\[
|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x - x_0|^{n+1}.
\]

In this case, \( n = 2, x_0 = 100, \) and \( x = 110 \) (the farthest from \( x_0 \) that we are considering).

\[
K_3 = \text{max of } |f'''(x)| \text{ on } [100, 110] = \text{max of } \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 110] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}
\]

Putting this all together, we have
\[
|f(x) - P_2(x)| \leq \frac{\frac{3}{800,000}}{3!} |110 - 100|^3 = \frac{1}{1600}.
\]

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \[
\int_{1}^{\infty} \frac{6 + \cos x}{x^{0.99}} \, dx
\]

For all \( x \geq 1 \), we have \( \frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \( \cos x \) is \( -1 \).

Since \( \int_{1}^{\infty} \frac{5}{x^{0.99}} \, dx \) diverges (compute yourself or notice that \( p = 0.99 < 1 \)), we know that the integral in question must diverge too.

(b) \[
\int_{1}^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx
\]

For all \( x \geq 1 \), we have \( \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2} \) (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[
4 \int_{1}^{\infty} \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_{1}^{t} \frac{dx}{x^2} = 4 \lim_{t \to \infty} \frac{-1}{x} \bigg|_{1}^{t} = 4 \lim_{t \to \infty} \left[ \frac{-1}{t} - \frac{-1}{1} \right] = 4[0 - (-1)] = 4
\]

Therefore, the original integral in question must converge to a value less than 4.
5. The probability density function (pdf) of the length (in minutes) of phone calls on a certain wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_0^t ke^{-0.2x} \, dx
\]

\[
= \lim_{t \to \infty} \left. \frac{ke^{-0.2x}}{-0.2} \right|_0^t
\]

\[
= \lim_{t \to \infty} \left. \frac{ke^{-0.2t}}{-0.2} \right|_0^t
\]

\[
= 0 - \frac{k}{-0.2} = 5k
\]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) **What fraction of calls last more than 3 minutes?**

\[
\int_3^\infty 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_3^t 0.2e^{-0.2x} \, dx
\]

\[
= \lim_{t \to \infty} \left. \frac{0.2e^{-0.2x}}{-0.2} \right|_3^t
\]

\[
= \lim_{t \to \infty} \left. -e^{-0.2t} - (-e^{-0.6}) \right|_3
\]

\[
= 0 + e^{-0.6}
\]

\[
= e^{-0.6} \approx 0.5488
\]

Note that we could instead have computed \( 1 - \int_0^3 0.2e^{-0.2x} \, dx \) and gotten the same answer, but the point of introducing pdf’s in this text seems to be to show how improper integrals are used.