INTEGRATION TIPS

- Substitution: usually let $u$ = a function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\int u \, dv = uv - \int v \, du$ or $\int uv' \, dx = uv - \int u'v \, dx$

How to choose which part is $u$? Let $u$ be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter $A$ is replaced by $P$, which stands for “polynomial.”)

<table>
<thead>
<tr>
<th>Logarithms (such as $\ln x$)</th>
<th>Inverse trig (such as $\arctan x, \arcsin x$)</th>
<th>Algebraic (such as $x, x^2, x^3 + 4$)</th>
<th>Trig (such as $\sin x, \cos 2x$)</th>
<th>Exponentials (such as $e^x, e^{3x}$)</th>
</tr>
</thead>
</table>

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

- Partial Fractions: here’s an illustrative example of the setup.

$$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>Substitution</th>
<th>$\sqrt{a^2 - x^2}$</th>
<th>$\sqrt{a^2 + x^2}$</th>
<th>$\sqrt{x^2 - a^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x = a \sin t$</td>
<td>$x = a \tan t$</td>
<td>$x = a \sec t$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sin^2 x$</th>
<th>$\cos^2 x$</th>
<th>$\sin^2 x + \cos^2 x = 1$</th>
<th>$\tan^2 x + 1 = \sec^2 x$</th>
<th>$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$</th>
<th>$\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$</th>
</tr>
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<tr>
<th>$\sin(2x)$</th>
<th>$2 \sin x \cos x$</th>
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- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>$\int \sin^m x \cos^n x , dx$</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd</td>
<td>Break off one factor of $\sin x$ and substitute $u = \cos x$.</td>
<td>$\sin^2 x = 1 - \cos^2 x$</td>
</tr>
<tr>
<td>$n$ odd</td>
<td>Break off one factor of $\cos x$ and substitute $u = \sin x$.</td>
<td>$\cos^2 x = 1 - \sin^2 x$</td>
</tr>
<tr>
<td>$m, n$ even</td>
<td>Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use integration by parts or identities shown to right of this box.</td>
<td>$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$</td>
</tr>
</tbody>
</table>
\[
\int \tan^m x \sec^n x \, dx
\]

<table>
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<tr>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sec x \tan x ) and substitute ( u = \sec x ).</td>
</tr>
<tr>
<td>( n ) even</td>
<td>Break off one factor of ( \sec^2 x ) and substitute ( u = \tan x ).</td>
</tr>
<tr>
<td>( m ) even, ( n ) odd</td>
<td>Use identity at right to reduce to powers of ( \sec x ) alone. Then use integration by parts or reduction formula (if allowed).</td>
</tr>
</tbody>
</table>

Useful Trigonometric Derivatives and Antiderivatives

\[
\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.
  
  \[
  \lim_{x \to \infty} e^x = \infty \\
  \lim_{x \to -\infty} e^{-x} = 0 \quad \text{Note: this is the same as } \lim_{x \to -\infty} e^x. \\
  \lim_{x \to \infty} 1/x = 0 \quad \text{Note: the answer is the same for } \lim_{x \to \infty} 1/x^2 \text{ and similar functions.} \\
  \lim_{x \to 0^+} 1/x = \infty \quad \text{Note: the answer is the same for } \lim_{x \to 0^+} 1/x^2 \text{ and similar functions.} \\
  \lim_{x \to \infty} \ln x = \infty \\
  \lim_{x \to 0^+} \ln x = -\infty \\
  \lim_{x \to \infty} \arctan x = \pi/2
\]

1. Evaluate the following.

(a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx = \int u^6 (1 - u^2) \, du = \int (u^6 - u^8) \, du = \frac{u^7}{7} - \frac{u^9}{9} + C = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]

Use \( \cos^2 x = 1 - \sin^2 x \).

(b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

\[
\frac{y}{10} = x \quad x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}} \\
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}} \\
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}} \\
= \int \sec t \, dt \\
= \ln |\sec t + \tan t| + C \\
\]
Now use \(1 + \tan^2 t = \sec^2 t\).
\[
= \ln \left| \sqrt{x^2 + 100} + \frac{x}{10} \right| + C \\
\]
Now use triangle above.
\[
\]
(c) This is an improper integral, so we need to use a limit.
\[
\int_3^\infty \frac{1}{x \ln x^{100}} \, dx = \lim_{t \to \infty} \int_3^t \frac{1}{x \ln x^{100}} \, dx \\
= \lim_{t \to \infty} \int_{x=3}^{x=t} \frac{1}{u^{100}} \, du \\
= \lim_{t \to \infty} \left[ \left. \frac{u^{-99}}{-99} \right|_{x=3}^{x=t} \right] \\
= \lim_{t \to \infty} \left[ \frac{-1}{99 (\ln t)^{99}} - \frac{-1}{99 (\ln 3)^{99}} \right] \\
= 0 - \frac{-1}{99 (\ln 3)^{99}} \\
= \frac{1}{99 (\ln 3)^{99}} \\
\]
So, the integral converges (to this value).
\[
\]
(d) We'll use integration by parts: \(u = x \Rightarrow du = dx\) and \(dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}\).
\[
\int_0^\infty xe^{-2x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ \left. xe^{-2x} \right|_{-2}^0 - \int_0^t e^{-2x} \, dx \right] \\
= \lim_{t \to \infty} \left[ \left. xe^{-2x} \right|_{-2}^0 - \frac{e^{-2x}}{4} \right]_0^t \\
= \lim_{t \to \infty} \left[ \frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}} \right]_0^t \\
= \lim_{t \to \infty} \left[ \frac{-t - 1}{2e^{2t}} - \frac{0 - 1}{4e^{2t}} \right] \\
= (0 - 0) - (0 - 1/4) \\
= 1/4 \\
\]
So, the integral converges (to this value).
\[
\]
(e) Partial Fractions:
\[
\text{Write } \frac{3x^2 + 2x - 13}{(x-3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3} \text{. Now multiply both sides by } (x-3)(x^2 + 1) \text{ to get}
\]
\[3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).\]

Let \(x = 3\). Then 20 = \(C(10)\), so \(C = 2\).

Let \(x = 0\). Then \(-13 = B(-3) + 2(1)\), so \(B = 5\).

Let \(x = 1\). Then \(-8 = (A(1) + 5)(-2) + 2(2)\), so \(A = 1\).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \(u = x^2 + 1\), so \(du = 2xdx\).

\[
= \ln u + \frac{5}{2} \arctan x + 2 \ln |x - 3| + D
\]

\[
= \frac{\ln(x^2 + 1)}{2} + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
\frac{4x^2 - 3x + 2}{x - 6} = \frac{4x^3 - 27x^2 + 20x - 17}{x - 6}
\]

\[
\frac{4x^3 - 24x^2}{-3x^2 + 18x}
\]

\[
\frac{2x}{2x - 12}
\]

\[
\frac{-5}{-5}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \(x = 1\) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{5} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to 1^+} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} \bigg|_{-1}^{a} + \lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} \bigg|_{b}^{5}
\]

\[
= \lim_{a \to 1^-} \left[ -\frac{1}{5(a - 1)^5} - \frac{-1}{5(-1 - 1)^5} \right] + \lim_{b \to 1^+} \left[ \frac{-1}{5(5 - 1)^5} - \frac{-1}{5(b - 1)^5} \right]
\]

Since \(\lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} = \infty\) and \(\lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} = \infty\), this integral diverges (to \(\infty\)).
2. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

\[
\begin{align*}
  f(x) &= x^{1/2} \\
  f'(x) &= \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\
  f''(x) &= -\frac{1}{4} x^{-3/2} = -\frac{1}{4x^{3/2}} \\
  f(100) &= 10 \\
  f'(100) &= \frac{1}{2\sqrt{100}} = \frac{1}{20} \\
  f''(100) &= -\frac{1}{4 \cdot 100^{3/2}} = -\frac{1}{4000}
\end{align*}
\]

\[
P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2
= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
\]

3. What is the maximum possible error that can occur in your Taylor approximation from
the previous problem on the interval \([100, 110]\)?

We know that \(|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1} \).

In this case, \(n = 2, x_0 = 100, \) and \(x = 110\) (the farthest from \(x_0\) that we are considering).

\[
K_3 = \text{max of } |f'''(x)| \text{ on } [100, 110] = \text{max of } \frac{3}{8x^{5/2}} \text{ on } [100, 110] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}
\]

Putting this all together, we have \(|f(x) - P_2(x)| \leq \frac{3}{3!} \cdot 110 - 100)^3 = \frac{1}{1600}\).

4. Use comparisons to show whether each of the following converges or diverges. If an
integral converges, also give a good upper bound for its value.

(a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

For all \(x \geq 1\), we have \( \frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \(\cos x\) is \(-1\).

Since \( \int_1^\infty \frac{5 \, dx}{x^{0.99}} \) diverges (compute yourself or notice that \(p = 0.99 < 1\)), we know that the integral in question must diverge too.

(b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

For all \(x \geq 1\), we have \( \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2} \). (We’ve made the denominator smaller and
the numerator larger, so the new fraction is larger.)

\[
4 \int_1^\infty \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2}
= 4 \lim_{t \to \infty} \left[ -\frac{1}{x} \right]_1^t
= 4 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{1}{1} \right]
= 4[0 - (-1)]
= 4
\]

Therefore, the original integral in question must converge to a value less than 4.
5. The probability density function (pdf) of the length (in minutes) of phone calls on a certain wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

[Students in the 8:00 section should omit this problem.]

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_{0}^{\infty} ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_{0}^{t} ke^{-0.2x} \, dx
\]

\[
= \lim_{t \to \infty} \left[ ke^{-0.2x} \right]_{0}^{t}
\]

\[
= \lim_{t \to \infty} \left( ke^{-0.2t} - ke^{0} \right) / -0.2
\]

\[
= 0 - k / -0.2
\]

\[
= 5k
\]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) **What fraction of calls last more than 3 minutes?**

\[
\int_{3}^{\infty} 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_{3}^{t} 0.2e^{-0.2x} \, dx
\]

\[
= \lim_{t \to \infty} \left[ 0.2e^{-0.2x} \right]_{3}^{t}
\]

\[
= \lim_{t \to \infty} -e^{-0.2t} - (-e^{-0.6})
\]

\[
= 0 + e^{-0.6}
\]

\[
= e^{-0.6} \approx 0.5488
\]

Note that we could instead have computed \( 1 - \int_{0}^{3} 0.2e^{-0.2x} \, dx \) and gotten the same answer.