Math 106: Review for Exam II - SOLUTIONS

INTEGRATION TIPS

• Substitution: usually let \( u = \) a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

• Parts: \[ \int u \, dv = uv - \int v \, du \quad \text{or} \quad \int uv' \, dx = uv - \int u'v \, dx \]

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below.

(The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial.”)

Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x, \arcsin x \))
Algebraic (such as \( x, x^2, x^3 + 4 \))
Trig (such as \( \sin x, \cos 2x \))
Exponentials (such as \( e^x, e^{3x} \))

• Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[ \frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5} \]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E)\) above it on the right.

• Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>( \sqrt{a^2 - x^2} )</th>
<th>( \sqrt{a^2 + x^2} )</th>
<th>( \sqrt{x^2 - a^2} )</th>
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<tbody>
<tr>
<td>Substitution</td>
<td>( x = a \sin t )</td>
<td>( x = a \tan t )</td>
<td>( x = a \sec t )</td>
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</table>

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 &= \sec^2 x \\
\cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) &= 2 \sin x \cos x
\end{align*}
\]

• Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
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<tbody>
<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m ) even AND ( n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use table of integrals #39–42 or identities shown to right of this box.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} ) or ( \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2} )</td>
</tr>
</tbody>
</table>
\[
\int \tan^m x \sec^n x \, dx
\]

| \(m\) odd | Break off one factor of \(\sec x \tan x\) and substitute \(u = \sec x\). | \(\tan^2 x = \sec^2 x - 1\) |
| \(n\) even | Break off one factor of \(\sec^2 x\) and substitute \(u = \tan x\). | \(\sec^2 x = \tan^2 x + 1\) |
| \(m\) even AND \(n\) odd | Use identity at right to reduce to powers of \(\sec x\) alone. Then use table of integrals \#51 or integration by parts. | \(\tan^2 x = \sec^2 x - 1\) |

**Useful Trigonometric Derivatives and Antiderivatives**

\[
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \sec x = \sec x \tan x \\
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \(\infty\) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\begin{align*}
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to \infty} e^{-x} &= 0 \\
\lim_{x \to \infty} 1/x &= 0 \quad \text{Note: this is the same as}\, \lim_{x \to -\infty} e^x. \\
\lim_{x \to \infty} 1/x &= \infty \quad \text{Note: the answer is the same for}\, \lim_{x \to 0^+} 1/x^2 \text{ and similar functions.} \\
\lim_{x \to \infty} \ln x &= \infty \\
\lim_{x \to 0^+} \ln x &= -\infty \\
\lim_{x \to \infty} \arctan x &= \pi/2
\end{align*}
\]

1. Evaluate the following.

(a) Let \(u = \sin x\), so \(du = \cos x \, dx\).

\[
\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \\
= \int u^6 (1 - u^2) \, du \\
= \int (u^6 - u^8) \, du \\
= \frac{u^7}{7} - \frac{u^9}{9} + C \\
= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
\]

(b) Let \(x = 10 \tan t\), so \(dx = 10 \sec^2 t \, dt\).

\[
x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
\sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
\tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}}
\]
\[
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}}
\]
\[
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}}
\]
\[
= \int \sec t \, dt
\]
\[
= \ln |\sec t + \tan t| + C
\]
Now use triangle above.

(c) This is an improper integral, so we need to use a limit.
\[
\int_{3}^{\infty} \frac{1}{x(\ln x)^{100}} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{1}{x(\ln x)^{100}} \, dx
\]
\[
= \lim_{t \to \infty} \int_{3}^{x=t} \frac{1}{u^{100}} \, du
\]
\[
= \lim_{t \to \infty} \frac{u^{-99}}{-99} \bigg|_{x=3}^{x=t}
\]
\[
= \lim_{t \to \infty} \left[ \frac{-1}{99(\ln x)^{99}} \bigg|_{3}^{t} \right]
\]
\[
= \lim_{t \to \infty} \left[ -\frac{1}{99(\ln t)^{99}} + \frac{1}{99(\ln 3)^{99}} \right]
\]
\[
= 0 - \frac{1}{99(\ln 3)^{99}}
\]
\[
= \frac{1}{99(\ln 3)^{99}}
\]
So, the integral converges (to this value).

(d) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2} \).
\[
\int_{0}^{\infty} xe^{-2x} \, dx = \lim_{t \to \infty} \int_{0}^{t} xe^{-2x} \, dx
\]
\[
= \lim_{t \to \infty} \left[ \frac{x e^{-2x}}{-2} \bigg|_{0}^{t} - \int_{0}^{t} \frac{e^{-2x}}{-2} \, dx \right]
\]
\[
= \lim_{t \to \infty} \left[ \frac{-xe^{-2x}}{-2} + e^{-2x} \bigg|_{0}^{t} \right]
\]
\[
= \lim_{t \to \infty} \left[ \frac{-xe^{-2x}}{-2} + e^{-2x} \bigg|_{0}^{t} \right]
\]
\[
= \lim_{t \to \infty} \left[ \frac{-t e^{-2t}}{2e^{2t}} - \frac{1}{4e^{2t}} \bigg|_{0}^{t} \right]
\]
\[
= \lim_{t \to \infty} \left[ \frac{-t e^{-2t}}{2e^{2t}} - \frac{1}{4e^{2t}} \bigg|_{0}^{t} \right]
\]
\[
= (0 - 0) - (0 - 1/4)
\]
\[
= 1/4
\]
So, the integral converges (to this value).

(e) Partial Fractions:
Write \( \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3} \). Now multiply both sides by \((x - 3)(x^2 + 1)\) to get
\[3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).\]

Let \(x = 3\). Then \(20 = C(10)\), so \(C = 2\).

Let \(x = 0\). Then \(-13 = B(-3) + 2(1)\), so \(B = 5\).

Let \(x = 1\). Then \(-8 = (A(1) + 5)(-2) + 2(2)\), so \(A = 1\).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

\[
= \int \left[ \frac{x}{x^2 + 1} + \frac{5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \(u = x^2 + 1\), so \(du = 2xdx\).

\[
= \ln \frac{x^2 + 1}{2} + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
x - 6 \quad \frac{4x^3 - 27x^2 + 20x - 17}{4x^3 - 24x^2 - 3x^2 + 18x}
\]

\[
\frac{2x}{5} - \frac{2x - 12}{5}
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \(x = 1\) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{5} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to 1^+} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to 1^-} \left[ -\frac{1}{5(a - 1)^5} \right]_{-1}^{a} + \lim_{b \to 1^+} \left[ -\frac{1}{5(b - 1)^5} \right]_{b}^{5}
\]

\[
= \lim_{a \to 1^-} \left[ -\frac{1}{5(a - 1)^5} \right] + \lim_{b \to 1^+} \left[ -\frac{1}{5(b - 1)^5} \right]
\]

Since \(\lim_{a \to 1^-} \frac{-1}{5(a - 1)^5} = \infty\) and \(\lim_{b \to 1^+} \frac{-1}{5(b - 1)^5} = \infty\), this integral diverges (to \(\infty\)).
2. Find the second-order Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \). Then use your polynomial to estimate \( \sqrt{105} \).

\[
\begin{align*}
f(x) &= x^{1/2} \\
f'(x) &= \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\
f''(x) &= -\frac{1}{4} x^{-3/2} = \frac{-1}{4x^{3/2}}
\end{align*}
\]

\[
P_2(x) = f(100) + f'(100)(x-100) + \frac{f''(100)}{2!} (x-100)^2
\]

\[
= 10 + \frac{x-100}{20} - \frac{(x-100)^2}{8000}
\]

Now, \( \sqrt{105} \approx P_2(105) = 10 + \frac{105-100}{20} - \frac{(105-100)^2}{8000} = 3279 \frac{3}{320} \).

3. What is the largest possible error that could have occurred in your estimate of \( \sqrt{105} \)? [Students in the 1:10 section may omit this problem.]

We know that \( |f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x-x_0|^{n+1} \).

In this case, \( n = 2, \ x_0 = 100, \) and \( x = 105 \).

\[K_3 = \text{max of } |f'''(x)| \text{ on } [100, 105] = \text{max of } \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 105] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}\]

Putting this all together, we have \( |f(x) - P_2(x)| \leq \frac{3}{320} \frac{1}{105 - 100}^3 = \frac{1}{12800} \).

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

For all \( x \geq 1 \), we have \( \frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \( \cos x \) is \(-1\).

Since \( \int_1^\infty \frac{5}{x^{0.99}} \, dx \) diverges (compute yourself or notice that \( p = 0.99 < 1 \)), we know that the integral in question must diverge too.

(b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

For all \( x \geq 1 \), we have \( \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^2} = 4 \frac{x}{x^2} = 4 \frac{1}{x} \) (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[4 \int_1^\infty \frac{dx}{x^2} = 4 \lim_{t \to \infty} \int_1^t \frac{dx}{x^2} = 4 \lim_{t \to \infty} \frac{-1}{x} \bigg|_1^t = 4 \lim_{t \to \infty} \frac{-1}{t} - \frac{-1}{1} = 4[0 - (-1)] = 4 \]

Therefore, the original integral in question must converge to a value less than 4.
(c) \( \int_0^1 \frac{9}{\sqrt{x^4 + x}} \, dx \)

For \( 0 < x \leq 1 \), we have \( \frac{9}{\sqrt{x^4 + x}} \leq \frac{9}{\sqrt{x}} = 9 \frac{1}{x^{1/2}} \). (We’ve made the denominator smaller, so the new fraction is larger.)

\[
9 \int_0^1 \frac{dx}{x^{1/2}} = 9 \lim_{t \to 0^+} \int_t^1 x^{-1/2} \, dx = 9 \lim_{t \to 0^+} \left[ \frac{1}{2} t^{1/2} \right]_t^1 = 9(2 - 0) = 18
\]

Therefore, the original integral in question must converge to a value less than 18.

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_0^t ke^{-0.2x} \, dx = \lim_{t \to \infty} \left[ \frac{ke^{-0.2x}}{-0.2} \right]_0^t = \lim_{t \to \infty} \left( ke^{-0.2t} \right) - \left( \frac{k}{-0.2} \right) = 0 - \frac{k}{-0.2} = \frac{5k}{1}
\]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) **What fraction of calls last more than 3 minutes?**

\[
\int_3^\infty 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_3^t 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \left[ \frac{0.2e^{-0.2x}}{-0.2} \right]_3^t = \lim_{t \to \infty} \left( -e^{-0.2t} + (-e^{-0.6}) \right) = 0 + e^{-0.6} = e^{-0.6} \approx 0.5488
\]

Note that we could instead have computed \( 1 - \int_0^3 0.2e^{-0.2x} \, dx \) and gotten the same answer.