INTEGRATION TIPS

- Substitution: usually let $u = a$ function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\int u \, dv = uv - \int v \, du$ or $\int uv' \, dx = uv - \int u'v \, dx$

How to choose which part is $u$? Let $u$ be the part that is higher up in the LIATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter $A$ is replaced by $P$, which stands for “polynomial”.)

Logarithms (such as $\ln x$)
Inverse trig (such as $\arctan x$, $\arcsin x$)
Algebraic (such as $x$, $x^2$, $x^3 + 4$)
Trig (such as $\sin x$, $\cos 2x$)
Exponentials (such as $e^x$, $e^{3x}$)

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

$$\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}$$

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>$\sqrt{a^2 - x^2}$</th>
<th>$\sqrt{a^2 + x^2}$</th>
<th>$\sqrt{x^2 - a^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitution</td>
<td>$x = a \sin t$</td>
<td>$x = a \tan t$</td>
<td>$x = a \sec t$</td>
</tr>
</tbody>
</table>

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$$
$$\sin(2x) = 2\sin x \cos x$$

$$\tan^2 x + 1 = \sec^2 x$$
$$\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>$\int \sin^m x \cos^n x , dx$</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd</td>
<td>Break off one factor of $\sin x$ and substitute $u = \cos x$.</td>
<td>$\sin^2 x = 1 - \cos^2 x$</td>
</tr>
<tr>
<td>$n$ odd</td>
<td>Break off one factor of $\cos x$ and substitute $u = \sin x$.</td>
<td>$\cos^2 x = 1 - \sin^2 x$</td>
</tr>
<tr>
<td>$m$ even AND $n$ even</td>
<td>Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use integration by parts or identities shown to right of this box.</td>
<td>$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$</td>
</tr>
<tr>
<td>$\int \tan^m x \sec^n x , dx$</td>
<td>Possible Strategy</td>
<td>Identity to Use</td>
</tr>
<tr>
<td>---------------------------------</td>
<td>------------------</td>
<td>----------------</td>
</tr>
<tr>
<td>$m$ odd</td>
<td>Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$.</td>
<td>$\tan^2 x = \sec^2 x - 1$</td>
</tr>
<tr>
<td>$n$ even</td>
<td>Break off one factor of $\sec^n x$ and substitute $u = \tan x$.</td>
<td>$\sec^2 x = \tan^2 x + 1$</td>
</tr>
<tr>
<td>$m$ even AND $n$ odd</td>
<td>Use identity at right to reduce to powers of $\sec x$ alone. Then use integration by parts or reduction formula (if allowed).</td>
<td>$\tan^2 x = \sec^2 x - 1$</td>
</tr>
</tbody>
</table>

Useful Trigonometric Derivatives and Antiderivatives

$$\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C$$

- Improper integrals: look for $\infty$ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

$$\lim_{x \to \infty} e^x = \infty$$
$$\lim_{x \to -\infty} e^{-x} = 0$$
Note: this is the same as $\lim_{x \to -\infty} e^x$

$$\lim_{x \to \infty} 1/x = 0$$
Note: the answer is the same for $\lim_{x \to \infty} 1/x^2$ and similar functions

$$\lim_{x \to 0^+} 1/x = \infty$$
Note: the answer is the same for $\lim_{x \to 0^+} 1/x^2$ and similar functions

$$\lim_{x \to \infty} \ln x = \infty$$

$$\lim_{x \to 0^+} \ln x = 0$$

$$\lim_{x \to \infty} \arctan x = \pi/2$$

1. Evaluate the following.

(a) $\int \sin^6 x \cos^3 x \, dx$

(b) $\int \frac{dx}{\sqrt{100 + x^2}}$
(c) $\int_3^\infty \frac{1}{x (\ln x)^{100}} \, dx$

(d) $\int_0^\infty x e^{-2x} \, dx$

(e) $\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx$

(f) $\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx$

(g) $\int_1^3 \frac{1}{x - 1} \, dx$
2. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

3. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval \([100, 110]\)?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

   (a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

   (b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

5. The probability density function (pdf) of the length (in minutes) of phone calls on a certain wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

   (a) What must be the value of \( k \)?

   (b) What fraction of calls last more than 3 minutes?