INTEGRATION TIPS

- **Substitution**: usually let $u$ = a function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.

- **Parts**: $\int u \, dv = uv - \int v \, du$ or $\int uv' \, dx = uv - \int u'v \, dx$

How to choose which part is $u$? Let $u$ be the part that is higher up in the ILATE mnemonic below. (The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter A is replaced by P, which stands for “polynomial.”)

- **Logarithms** (such as $\ln x$)
- **Inverse trig** (such as $\arctan x$, $\arcsin x$)
- **Algebraic** (such as $x$, $x^2$, $x^3 + 4$)
- **Trig** (such as $\sin x$, $\cos 2x$)
- **Exponentials** (such as $e^x$, $e^{3x}$)

- **Rational Functions** (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

- **Partial Fractions**: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x - 3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term $(Dx + E)$ here above it on the right.

- **Trigonometric Substitutions**: some suggested substitutions and useful formulae follow.

\[
\begin{array}{|c|c|c|}
\hline
\text{Radical Form} & \sqrt{a^2 - x^2} & \sqrt{a^2 + x^2} & \sqrt{x^2 - a^2} \\
\hline
\text{Substitution} & x = a \sin t & x = a \tan t & x = a \sec t \\
\hline
\end{array}
\]

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} \\
\tan^2 x + 1 &= \sec^2 x \\
\cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2} \\
\sin(2x) &= 2 \sin x \cos x \\
\end{align*}
\]

- **Powers of Trigonometric Functions**: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>$\int \sin^m x \cos^n x , dx$</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd</td>
<td>Break off one factor of $\sin x$ and substitute $u = \cos x$.</td>
<td>$\sin^2 x = 1 - \cos^2 x$</td>
</tr>
<tr>
<td>$n$ odd</td>
<td>Break off one factor of $\cos x$ and substitute $u = \sin x$.</td>
<td>$\cos^2 x = 1 - \sin^2 x$</td>
</tr>
<tr>
<td>$m$ even AND $n$ even</td>
<td>Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use table of integrals #39–42</td>
<td>$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$ or $\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$</td>
</tr>
</tbody>
</table>
\[
\int \tan^m x \sec^n x \, dx
\]

<table>
<thead>
<tr>
<th>Possible Strategy</th>
<th>Identity to Use</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd</td>
<td>( \tan^2 x = \sec^2 x - 1 )</td>
</tr>
<tr>
<td>$n$ even</td>
<td>( \sec^2 x = \tan^2 x + 1 )</td>
</tr>
<tr>
<td>$m$ even AND $n$ odd</td>
<td>Use identity at right to reduce to powers of $\sec x$ alone. Then use table of integrals #51 or integration by parts.</td>
</tr>
<tr>
<td></td>
<td>( \sec^2 x = \sec^2 x - 1 )</td>
</tr>
</tbody>
</table>

**Useful Trigonometric Derivatives and Antiderivatives**

\[
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \sec x = \sec x \tan x \\
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\begin{align*}
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to \infty} e^{-x} &= 0 \\
\lim_{x \to \infty} \frac{1}{x} &= 0 \\
\lim_{x \to \infty} \frac{1}{x^2} &= 0 \\
\lim_{x \to \infty} \ln x &= \infty \\
\lim_{x \to 0^+} \ln x &= -\infty
\end{align*}
\]

1. **Evaluate the following.**

   (a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

   \[
   \int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx
   \]

   \[
   = \int u^6 (1 - u^2) \, du
   \]

   \[
   = \int (u^6 - u^8) \, du
   \]

   \[
   = \frac{u^7}{7} - \frac{u^9}{9} + C
   \]

   \[
   = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
   \]

   Use \( \cos^2 x = 1 - \sin^2 x \).

   (b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

   \[
   \begin{array}{c}
   y \\
   \square \bigtriangleup \\
   10 \\
   x
   \end{array}
   \]

   \[
   x^2 + 10^2 = y^2 \Rightarrow y = \sqrt{x^2 + 100}
   \]

   \[
   \sec t = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10}
   \]

   \[
   \tan t = \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
   \]
\[ \int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \ dt}{\sqrt{100 + 100 \tan^2 t}} \]
\[ = \int \frac{10 \sec^2 t \ dt}{10 \sqrt{1 + \tan^2 t}} \]
\[ = \int \frac{\sec^2 t \ dt}{\sqrt{\sec^2 t}} \]
\[ = \int \sec t \ dt \]
\[ = \ln |\sec t + \tan t| + C \]

Now use triangle above.

\[ = \ln \left| \frac{\sqrt{x^2 + 100}}{10} + \frac{x}{10} \right| + C \]

(c) This is an improper integral, so we need to use a limit.

\[ \int_3^\infty \frac{1}{x(\ln x)^{100}} \ dx = \lim_{t \to \infty} \int_3^t \frac{1}{x(\ln x)^{100}} \ dx \]
\[ = \lim_{t \to \infty} \int_{x=3}^{x=t} \frac{1}{u^{100}} \ du \]
\[ = \lim_{t \to \infty} \left[ \frac{u^{-99}}{-99} \right]_{x=3}^{x=t} \]
\[ = \lim_{t \to \infty} \left[ \frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \]
\[ = 0 - \frac{-1}{99(\ln 3)^{99}} \]
\[ = \frac{1}{99(\ln 3)^{99}} \]

So, the integral converges (to this value).

(d) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2} \).

\[ \int_0^\infty xe^{-2x} \ dx = \lim_{t \to \infty} \int_0^t xe^{-2x} \ dx \]
\[ = \lim_{t \to \infty} \left[ \frac{-x}{2e^{2x}} \right]_0^t - \int_0^t \frac{e^{-2x}}{-2} \ dx \]
\[ = \lim_{t \to \infty} \left[ \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} \right]_0^t \]
\[ = \lim_{t \to \infty} \left[ \frac{-t}{2e^{2t}} - \frac{1}{4e^{2t}} \right] - \left[ \frac{0}{2e^0} - \frac{1}{4e^0} \right] \]
\[ = 0 - (0 - 1/4) \]
\[ = 1/4 \]

So, the integral converges (to this value).

(e) Partial Fractions:
Write
\[ \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3} \]

Now multiply both sides by \((x - 3)(x^2 + 1)\) to get
\[3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1).\]

Let \(x = 3\). Then \(20 = C(10)\), so \(C = 2\).
Let \(x = 0\). Then \(-13 = B(-3) + 2(1)\), so \(B = 5\).
Let \(x = 1\). Then \(-8 = (A(1) + 5)(-2) + 2(2)\), so \(A = 1\).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \(u = x^2 + 1\), so \(du = 2x \, dx\).

\[
= \int \left[ \frac{1}{u} + \frac{2}{x - 3} \right] \, dx = \ln u + 5 \arctan x + 2 \ln |x - 3| + D
\]

\[
= \ln(x^2 + 1) + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

\[
x - 6 \quad 4x^3 - 27x^2 + 20x - 17
\]

\[
= x - 6 \quad 4x^3 - 24x^2 - 3x^2 + 18x
\]

\[
= 2x \quad 2x - 12
\]

\[
= -5
\]

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \(x = 1\) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_{-1}^{b} \frac{1}{(x - 1)^6} \, dx = \int_{-1}^{1} \frac{dx}{(x - 1)^6} + \int_{1}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to -1^-} \int_{-1}^{a} \frac{dx}{(x - 1)^6} + \lim_{b \to +1^-} \int_{b}^{5} \frac{dx}{(x - 1)^6}
\]

\[
= \lim_{a \to -1^-} \left[ \frac{-1}{5(a - 1)^5} \right]_{-1}^{a} + \lim_{b \to +1^-} \left[ \frac{-1}{5(b - 1)^5} \right]_{b}^{5}
\]

\[
= \lim_{a \to -1^-} \left[ \frac{-1}{5(a - 1)^5} - \frac{-1}{5(-1 - 1)^5} \right] + \lim_{b \to +1^-} \left[ \frac{-1}{5(5 - 1)^5} - \frac{-1}{5(1 - 1)^5} \right]
\]

Since \(\lim_{a \to -1^-} \frac{-1}{5(a - 1)^5} = \infty\) and \(\lim_{b \to +1^-} \frac{-1}{5(b - 1)^5} = \infty\), this integral diverges (to \(\infty\)).
2. Find the second-order Taylor polynomial for \( f(x) = \sqrt{x} \) based at \( x_0 = 100 \). Then use your polynomial to estimate \( \sqrt{105} \).

\[
\begin{align*}
f(x) &= x^{1/2} \\
f'(x) &= \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \\
f''(x) &= -\frac{1}{4} x^{-3/2} = -\frac{1}{4x^{3/2}}
\end{align*}
\]

\[
P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2
\]

\[
= 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}
\]

Now, \( \sqrt{105} \approx P_2(105) = 10 + \frac{105 - 100}{20} - \frac{(105 - 100)^2}{8000} = \frac{3279}{320} \).

3. What is the largest possible error that could have occurred in your estimate of \( \sqrt{105} \)?

We know that \( |f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1} \).

In this case, \( n = 2, x_0 = 100, \) and \( x = 105 \).

\[
K_3 = \max |f'''(x)| \text{ on } [100, 105] = \max \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 105] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000}
\]

Putting this all together, we have \( |f(x) - P_2(x)| \leq \frac{3}{800,000} |105 - 100|^3 = \frac{1}{12800} \).

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \( \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \)

For all \( x \geq 1 \), we have \( \frac{6 + \cos x}{x^{0.99}} \geq \frac{6 - 1}{x^{0.99}} = \frac{5}{x^{0.99}} \) because the minimum value of \( \cos x \) is \(-1\).

Since \( \int_1^\infty \frac{5}{x^{0.99}} \, dx \) diverges (compute yourself or notice that \( p = 0.99 < 1 \)), we know that the integral in question must diverge too.

(b) \( \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

For all \( x \geq 1 \), we have \( \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = 4 \frac{1}{x^2} \). (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[
4 \int_1^\infty x^{-2} \, dx = 4 \lim_{t \to \infty} \int_1^t x^{-2} \, dx \\
= 4 \lim_{t \to \infty} \left[ -\frac{1}{t} \right]_1^t \\
= 4 \lim_{t \to \infty} \left[ -\frac{1}{t} - \frac{1}{1} \right] \\
= 4[0 - (-1)] \\
= 4
\]

Therefore, the original integral in question must converge to a value less than 4.
5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty ke^{-0.2x} \, dx = \lim_{t \to \infty} \int_0^t ke^{-0.2x} \, dx \\
= \lim_{t \to \infty} \left. \frac{k e^{-0.2x}}{-0.2} \right|_0^t \\
= \lim_{t \to \infty} \left( -0.2 \cdot \frac{k e^t}{-0.2} \right) - \left( -0.2 \cdot \frac{k e^0}{-0.2} \right) \\
= 0 - \frac{k}{-0.2} \\
= 5k
\]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) **What fraction of calls last more than 3 minutes?**

\[
\int_3^\infty 0.2e^{-0.2x} \, dx = \lim_{t \to \infty} \int_3^t 0.2e^{-0.2x} \, dx \\
= \lim_{t \to \infty} \left. \frac{0.2e^{-0.2x}}{-0.2} \right|_3^t \\
= \lim_{t \to \infty} \left( -0.2 \cdot \frac{e^{-0.2t}}{-0.2} \right) - \left( -0.2 \cdot \frac{e^{-0.6}}{-0.2} \right) \\
= 0 + e^{-0.6} \\
= e^{-0.6} \approx 0.5488
\]

Note that we could instead have computed \( 1 - \int_0^3 0.2e^{-0.2x} \, dx \) and gotten the same answer.