INTEGRATION TIPS

- Substitution: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below.

(Logarithms (such as \( \ln x \))
Inverse trig (such as \( \arctan x, \arcsin x \))
Algebraic (such as \( x, x^2, x^3 + 4 \))
Trig (such as \( \sin x, \cos 2x \))
Exponentials (such as \( e^x, e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)^2(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E)\) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>( \sqrt{a^2 - x^2} )</th>
<th>( \sqrt{a^2 + x^2} )</th>
<th>( \sqrt{x^2 - a^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Substitution</td>
<td>( x = a \sin t )</td>
<td>( x = a \tan t )</td>
<td>( x = a \sec t )</td>
</tr>
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</table>

\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1 \\
\sin^2 x &= \frac{1}{2} - \frac{\cos(2x)}{2} \\
\sin(2x) &= 2 \sin x \cos x
\end{align*}
\]

\[
\begin{align*}
\tan^2 x + 1 &= \sec^2 x \\
\cos^2 x &= \frac{1}{2} + \frac{\cos(2x)}{2}
\end{align*}
\]

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
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<tbody>
<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m ) even AND ( n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use table of integrals #39–42 or identities shown to right of this box.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} )</td>
</tr>
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</table>
\[ \int \tan^m x \sec^n x \, dx \quad \text{Possible Strategy} \quad \text{Identity to Use} \]

| \( m \) odd | Break off one factor of \( \sec x \tan x \) and substitute \( u = \sec x \). | \( \tan^2 x = \sec^2 x - 1 \) |
| \( n \) even | Break off one factor of \( \sec^2 x \) and substitute \( u = \tan x \). | \( \sec^2 x = \tan^2 x + 1 \) |
| \( m \) even AND \( n \) odd | Use identity at right to reduce to powers of \( \sec x \) alone. Then use table of integrals #51 or integration by parts. | \( \tan^2 x = \sec^2 x - 1 \) |

Useful Trigonometric Derivatives and Antiderivatives

\[ \frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \sec x = \sec x \tan x \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C \]

- Improper integrals: look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[ \lim_{x \to \infty} e^x = \infty \]
\[ \lim_{x \to \infty} e^{-x} = 0 \]
Note: this is the same as \( \lim_{x \to -\infty} e^x \)

\[ \lim_{x \to \infty} \frac{1}{x} = 0 \]
Note: the answer is the same for \( \lim_{x \to \infty} \frac{1}{x^2} \) and similar functions

\[ \lim_{x \to -0^+} \frac{1}{x} = -\infty \]
Note: the answer is the same for \( \lim_{x \to 0^-} \frac{1}{x^2} \) and similar functions

\[ \lim_{x \to \infty} \ln x = \infty \]
\[ \lim_{x \to -0^+} \ln x = -\infty \]

1. Evaluate the following.

(a) \( \int \sin^6 x \cos^3 x \, dx \)

(b) \( \int \frac{dx}{\sqrt{100 + x^2}} \)
(c) \[ \int_3^\infty \frac{1}{x(\ln x)^{100}} \, dx \]

(d) \[ \int_0^\infty xe^{-2x} \, dx \]

(e) \[ \int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx \]

(f) \[ \int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx \]

(g) \[ \int_{-1}^5 \frac{1}{(x - 1)^6} \, dx \]
2. Find the second-order Taylor polynomial for \( f(x) = \sqrt{x} \) based at \( x_0 = 100 \). Then use your polynomial to estimate \( \sqrt{105} \).

3. What is the largest possible error that could have occurred in your estimate of \( \sqrt{105} \)?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.
   
   (a) \( \int_{1}^{\infty} \frac{6 + \cos x}{x^{0.99}} \, dx \)

   (b) \( \int_{1}^{\infty} \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \)

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by \( f(x) = ke^{-0.2x} \) where \( x \) is the number of minutes. Note that the domain is \( x \geq 0 \) since we can’t have a negative number of minutes.

   (a) What must be the value of \( k \)?

   (b) What fraction of calls last more than 3 minutes?