INTEGRATION TIPS

- Substitution: usually let \( u = \) a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.

- Parts: \( \int u \, dv = uv - \int v \, du \) or \( \int uv' \, dx = uv - \int u'v \, dx \)

How to choose which part is \( u \)? Let \( u \) be the part that is higher up in the LIATE mnemonic below.

(The mnemonics ILATE and LIPET will work equally well if you have learned one of those instead; in the latter \( A \) is replaced by \( P \), which stands for “polynomial”.)

Logarithms (such as \( \ln x \))

Inverse trig (such as \( \arctan x, \arcsin x \))

Algebraic (such as \( x, x^2, x^3 + 4 \))

Trig (such as \( \sin x, \cos 2x \))

Exponentials (such as \( e^x, e^{3x} \))

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

Partial Fractions: here’s an illustrative example of the setup.

\[
\frac{3x^2 + 11}{(x + 1)(x - 3)(x^2 + 5)} = \frac{A}{x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 5}
\]

Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor \((x - 3)\) on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term \((Dx + E\) here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

<table>
<thead>
<tr>
<th>Radical Form</th>
<th>( \sqrt{a^2 - x^2} )</th>
<th>( \sqrt{a^2 + x^2} )</th>
<th>( \sqrt{x^2 - a^2} )</th>
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</thead>
<tbody>
<tr>
<td>Substitution</td>
<td>( x = a \sin t )</td>
<td>( x = a \tan t )</td>
<td>( x = a \sec t )</td>
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</tbody>
</table>

\[
\sin^2 x + \cos^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x \quad \cos^2 x = 1 + \frac{\cos(2x)}{2} \quad \sin(2x) = 2 \sin x \cos x
\]

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

<table>
<thead>
<tr>
<th>( \int \sin^m x \cos^n x , dx )</th>
<th>Possible Strategy</th>
<th>Identity to Use</th>
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<tbody>
<tr>
<td>( m ) odd</td>
<td>Break off one factor of ( \sin x ) and substitute ( u = \cos x ).</td>
<td>( \sin^2 x = 1 - \cos^2 x )</td>
</tr>
<tr>
<td>( n ) odd</td>
<td>Break off one factor of ( \cos x ) and substitute ( u = \sin x ).</td>
<td>( \cos^2 x = 1 - \sin^2 x )</td>
</tr>
<tr>
<td>( m, n ) even</td>
<td>Use ( \sin^2 x + \cos^2 x = 1 ) to reduce to only powers of ( \sin x ) or only powers of ( \cos x ), then use integration by parts or identities shown to right of this box.</td>
<td>( \sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} )</td>
</tr>
</tbody>
</table>
\( \int \tan^m x \sec^n x \, dx \)

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<tr>
<td>Break off one factor of ( \sec x \tan x ) and substitute ( u = \sec x ).</td>
<td>( \tan^2 x = \sec^2 x - 1 )</td>
</tr>
<tr>
<td>Break off one factor of ( \sec^2 x ) and substitute ( u = \tan x ).</td>
<td>( \sec^2 x = \tan^2 x + 1 )</td>
</tr>
<tr>
<td>Use identity at right to reduce to powers of ( \sec x ) alone. Then use integration by parts or reduction formula (if allowed).</td>
<td>( \tan^2 x = \sec^2 x - 1 )</td>
</tr>
</tbody>
</table>

**Useful Trigonometric Derivatives and Antiderivatives**

\[
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \sec x = \sec x \tan x \\
\int \sec x \, dx = \ln |\sec x + \tan x| + C
\]

**Improper integrals:** look for \( \infty \) as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

\[
\begin{align*}
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to \infty} e^{-x} &= 0 \\
\lim_{x \to \infty} \frac{1}{x} &= 0 \\
\lim_{x \to 0^+} \ln x &= \infty \\
\lim_{x \to 0^+} \ln x &= -\infty \\
\lim_{x \to \infty} \arctan x &= \frac{\pi}{2}
\end{align*}
\]

1. **Evaluate the following.**

   (a) Let \( u = \sin x \), so \( du = \cos x \, dx \).

   \[
   \int \sin^6 x \cos^3 x \, dx = \int \sin^6 x (1 - \sin^2 x) \cos x \, dx
   = \int u^6 (1 - u^2) \, du
   = \int \left( u^6 - u^8 \right) \, du
   = \frac{u^7}{7} - \frac{u^9}{9} + C
   = \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C
   \]

   Use \( \cos^2 x = 1 - \sin^2 x \).

   (b) Let \( x = 10 \tan t \), so \( dx = 10 \sec^2 t \, dt \).

   \[
   y
   \]

   \[
   \begin{align*}
x^2 + 10^2 &= y^2 \Rightarrow y = \sqrt{x^2 + 100} \\
\sec t &= \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{x^2 + 100}}{10} \\
\tan t &= \frac{\text{opp}}{\text{adj}} = \frac{x}{10}
\end{align*}
\]
\[
\int \frac{dx}{\sqrt{100 + x^2}} = \int \frac{10 \sec^2 t \, dt}{\sqrt{100 + 100 \tan^2 t}} \\
= \int \frac{10 \sec^2 t \, dt}{10 \sqrt{1 + \tan^2 t}} \\
= \int \frac{\sec^2 t \, dt}{\sqrt{\sec^2 t}} \\
= \int \frac{\sec t \, dt}{\sqrt{\sec^2 t}} \\
= \ln |\sec t + \tan t| + C \\
= \ln \left| \frac{\sqrt{x^2 + 100} + x}{10} \right| + C \\
\]

Now use \(1 + \tan^2 t = \sec^2 t\).

Now use triangle above.

\(c\) This is an improper integral, so we need to use a limit.

\[
\int_{3}^{\infty} \frac{1}{x(\ln x)^{100}} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{1}{x(\ln x)^{100}} \, dx \\
= \lim_{t \to \infty} \int_{3}^{t} \frac{1}{u^{100}} \, du \\
= \lim_{t \to \infty} \left[ \frac{-1}{99(\ln x)^{99}} \right]_{3}^{t} \\
= \lim_{t \to \infty} \left[ \frac{-1}{99(\ln t)^{99}} - \frac{-1}{99(\ln 3)^{99}} \right] \\
= \left[ \frac{1}{99(\ln 3)^{99}} \right] \\
\]

So, the integral converges (to this value).

\(d\) We’ll use integration by parts: \(u = x \Rightarrow du = dx\) and \(dv = e^{-2x} \Rightarrow v = \frac{e^{-2x}}{-2}\).

\[
\int_{0}^{\infty} xe^{-2x} \, dx = \lim_{t \to \infty} \int_{0}^{t} xe^{-2x} \, dx \\
= \lim_{t \to \infty} \left[ \frac{x e^{-2x}}{-2} \right]_{0}^{t} - \int_{0}^{t} \frac{e^{-2x}}{-2} \, dx \\
= \lim_{t \to \infty} \left[ \frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4} \right]_{0}^{t} \\
= \lim_{t \to \infty} \left[ \frac{-xe^{-2x}}{2e^{2t}} + \frac{1}{4e^{2t}} \right]_{0}^{t} \\
= \lim_{t \to \infty} \left[ \frac{-t - \frac{1}{4e^{2t}} - \frac{0}{2e^{0}} - \frac{1}{4e^{0}}} {2e^{2t}} \right] \\
= (0 - 0) - (0 - 1/4) \\
= 1/4 \\
\]

So, the integral converges (to this value).

\(e\) Partial Fractions:

Write \(\frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 3}\). Now multiply both sides by \((x - 3)(x^2 + 1)\) to get
\[ 3x^2 + 2x - 13 = (Ax + B)(x - 3) + C(x^2 + 1). \]

Let \( x = 3 \). Then \( 20 = C(10) \), so \( C = 2 \).
Let \( x = 0 \). Then \( -13 = B(-3) + 2(1) \), so \( B = 5 \).
Let \( x = 1 \). Then \( -8 = (A(1) + 5)(-2) + 2(2) \), so \( A = 1 \).

\[
\int \frac{3x^2 + 2x - 13}{(x - 3)(x^2 + 1)} \, dx = \int \left[ \frac{x + 5}{x^2 + 1} + \frac{2}{x - 3} \right] \, dx
\]

Let \( u = x^2 + 1 \), so \( du = 2xdx \).

\[
\int \frac{4x^2 - 3x + 2}{x - 6} \, dx = \int \left[ \frac{4x^2 - 27x^2 + 20x - 17}{4x^3 - 24x^2} \right] \, dx
\]

\[
= \frac{4x^3}{x - 6} - 3x^2 + 18x
\]

\[
+ \frac{2x}{x - 6}
\]

\[
- \frac{2x - 12}{-5} = \ln(x^2 + 1) + 5 \arctan x + 2 \ln |x - 3| + D
\]

(f) Since the degree of the numerator is greater than or equal to the degree of the denominator, we do long division.

Now, we compute the integral.

\[
\int \frac{4x^3 - 27x^2 + 20x - 17}{x - 6} \, dx = \int \left[ 4x^2 - 3x + 2 - \frac{5}{x - 6} \right] \, dx = \frac{4x^3}{3} - \frac{3x^2}{2} + 2x - 5 \ln |x - 6| + C
\]

(g) This integral is improper at \( x = 1 \) because the integrand has a vertical asymptote there.

\[
\int_1^3 \frac{1}{x - 1} \, dx = \lim_{t \to 1^+} \int_1^3 \frac{1}{x - 1} \, dx = \lim_{t \to 1^+} \ln |x - 1|^3
\]

Since \( \lim_{t \to 1^+} (-\ln |t - 1|) = \infty \), this integral diverges (to \( \infty \)).

2. Find the second-degree Taylor polynomial for \( f(x) = \sqrt{x} \) centered at \( x = 100 \).

\[
f(x) = x^{1/2} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad f(100) = 10
\]

\[
f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \quad \quad \quad \quad \quad \quad \quad \quad \quad f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}
\]

\[
f''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}} \quad \quad \quad \quad \quad \quad \quad \quad \quad f''(100) = \frac{-1}{4 \cdot 100^{3/2}} = \frac{-1}{4000}
\]
\[ P_2(x) = f(100) + f'(100)(x - 100) + \frac{f''(100)}{2!}(x - 100)^2 \]
\[ = 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000} \]

3. What is the maximum possible error that can occur in your Taylor approximation from the previous problem on the interval [100, 110]?

We know that \(|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}.

In this case, \(n = 2\), \(x_0 = 100\), and \(x = 110\) (the farthest from \(x_0\) that we are considering).

\[ K_3 = \text{max of } |f'''(x)| \text{ on } [100, 110] = \text{max of } \left| \frac{3}{8x^{5/2}} \right| \text{ on } [100, 110] = \frac{3}{8 \cdot 100^{5/2}} = \frac{3}{800,000} \]

Putting this all together, we have \(|f(x) - P_2(x)| \leq \frac{3}{3!} \cdot 110 - 100^3 = \frac{1}{1600}.

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

(a) \[ \int_1^\infty \frac{6 + \cos x}{x^{0.99}} \, dx \]

For all \(x \geq 1\), we have \(\frac{6 + \cos x}{x^{0.99}} \geq 6 - 1 = 5\) because the minimum value of \(\cos x\) is \(-1\).

Since \[\int_1^\infty \frac{5}{x^{0.99}} \, dx \text{ diverges (compute yourself or notice that } p = 0.99 < 1\text{)}, we know that the integral in question must diverge too.

(b) \[ \int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \, dx \]

For all \(x \geq 1\), we have \(\frac{4x^3 - 2x^2}{2x^4 + x^5 + 1} \leq \frac{4x^3}{x^5} = \frac{4}{x^2}\). (We’ve made the denominator smaller and the numerator larger, so the new fraction is larger.)

\[ 4 \int_1^\infty dx = 4 \lim_{t \to \infty} \int_1^t dx \]
\[ = 4 \lim_{t \to \infty} \left[ \frac{1}{1} \right]^t \]
\[ = 4 \lim_{t \to \infty} \left( \frac{1}{t} - \frac{1}{1} \right) \]
\[ = 4[0 - (-1)] \]
\[ = 4 \]

Therefore, the original integral in question must converge to a value less than 4.

5. The probability density function (pdf) of the length (in minutes) of phone calls on a certain wireless network is given by \(f(x) = ke^{-0.2x}\) where \(x\) is the number of minutes. Note that the domain is \(x \geq 0\) since we can’t have a negative number of minutes.

(a) What must be the value of \(k\)?
We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[ \int_0^\infty k e^{-0.2x} \, dx = \lim_{t \to \infty} \int_0^t k e^{-0.2x} \, dx \]

\[ = \lim_{t \to \infty} \left[ \frac{k e^{-0.2x}}{-0.2} \right]_0^t \]

\[ = \lim_{t \to \infty} \left( \frac{k e^{-0.2t}}{-0.2} - \frac{k e^0}{-0.2} \right) \]

\[ = 0 - \frac{k}{-0.2} \]

\[ = 5k \]

So, we have \( 5k = 1 \) or \( k = 0.2 \).

(b) **What fraction of calls last more than 3 minutes?**

\[ \int_3^\infty 0.2 e^{-0.2x} \, dx = \lim_{t \to \infty} \int_3^t 0.2 e^{-0.2x} \, dx \]

\[ = \lim_{t \to \infty} \left( \frac{0.2 e^{-0.2x}}{-0.2} \right) \bigg|_3^t \]

\[ = \lim_{t \to \infty} \left( \frac{e^{-0.2t}}{-0.2} - \frac{e^{-0.6}}{-0.2} \right) \]

\[ = 0 + e^{-0.6} \]

\[ = e^{-0.6} \approx 0.5488 \]

Note that we could instead have computed \( 1 - \int_0^3 0.2 e^{-0.2x} \, dx \) and gotten the same answer, but the point of introducing pdf’s in this text seems to be to show how improper integrals are used.