Math 106 - Quiz 6 - October 27, 2006

Instructions: Show all of your work and circle your final answers. Calculators are allowed, but notes and books are not.

1. Find the Taylor polynomial of degree 3 for \( f(x) = \ln x \) at \( x_0 = 1 \).

2. Use your answer in the previous problem to find an estimate for \( \ln 2 \). (Your final answer should be a (single) fraction.)

3. According to Taylor's Theorem, what is the maximum possible error that could occur with this Taylor polynomial on the interval \([1, 2]\)?

\[
\begin{align*}
P_3(x) &= f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \frac{f'''(x_0)}{3!} (x-x_0)^3 \\
P_3(x) &= \ln 1 + \frac{1}{1}(x-1) + \frac{-1}{2}(x-1)^2 + \frac{2}{3}(x-1)^3 \\
P_3(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3.
\end{align*}
\]

(2)

\[
P_3(x) \approx f(x) \text{ for } x \text{-values near } x_0 = 1. \text{ So } f(2) \approx P_3(2). \text{ And } f(2) = \ln 2.
\]

\[
\begin{align*}
P_3(2) &= (2-1) - \frac{1}{2}(2-1)^2 + \frac{1}{3}(2-1)^3 \\
&= 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} - \frac{2}{6} + \frac{2}{6} = \frac{5}{6} \approx \ln 2.
\end{align*}
\]

(3)

Max error? \( |f(x) - P_3(x)| \leq \frac{K_4}{4!} |x-1|^4 \), where \( |f^{(n)}(x)| \leq K_4 \text{ for all } x \text{ in } [1, 2] \).

\[
f^{(n)}(x) = -(\ln x)^{n-1} \cdot -\frac{1}{x^n}.
\]

Among \( x \)-values in \([1, 2]\), this is largest when \( x = 1 \). So \( K_4 = \frac{6}{3} = 2 \) is a valid upper bound.

Then \( |f(x) - P_3(x)| \leq \frac{6}{24} |x-1|^4 = \frac{1}{4} |x-1|^4 \). Among \( x \)-values in \([1, 2]\), this is largest when \( x = 2 \). So \( |f(x) - P_3(x)| \leq \frac{1}{4} |x-1|^4 = \frac{1}{4} \). Thus, max error is \( \frac{1}{4} \).