(a) Find the third-order Taylor polynomial $P_3(x)$ for the function $f(x) = \sqrt[3]{x}$ based at $x_0 = 1$. Use your result to approximate the value $\sqrt[3]{2}$.

Since $f(x) = \sqrt[3]{x} = x^{1/3}$. It follows that $f'(x) = \frac{1}{3}x^{-2/3}$, $f''(x) = -\frac{2}{9}x^{-5/3}$, and $f'''(x) = \frac{10}{27}x^{-8/3}$. At $x = 1$, we have $f(1) = 1$, $f'(1) = \frac{1}{3}$, $f''(1) = -\frac{2}{9}$, $f'''(1) = \frac{10}{27}$. The third-order Taylor polynomial based at $x_0 = 1$ is given by

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$= 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3.$$  

We estimate $f(2) = \sqrt[3]{2}$ by $P_3(2) = 1 + \frac{1}{3} - \frac{1}{9} + \frac{5}{81} = \frac{104}{81}$.

(b) Determine whether the following improper integral converges. If it does, find its value.

$$\int_0^1 \frac{dx}{(x-1)^{2/3}}$$

First note that this integral is improper at $x = 1$ so that (by definition) we have

$$\int_0^1 \frac{dx}{(x-1)^{2/3}} = \lim_{b \to 1} \int_0^b \frac{dx}{(x-1)^{2/3}}$$

$$= \lim_{b \to 1} \frac{(x-1)^{1/3}}{1/3} \bigg|_0^b$$

$$= \lim_{b \to 1} 3(b - 1)^{1/3} - 3(-1)^{1/3} = 0 - (-3) = 3.$$