TEST 1

Math 205
10/10/11

Name: ____________________________

by writing my name i swear by the honor code

____________________________________

Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can be reduced into RREF with calculator without showing steps). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).

- Circle or otherwise indicate your final answers.

- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements. Put a smiley face next to your name for one point.

- This test has 8 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!

- Good luck!
1. (13 points) Consider the system of equations:

\[
\begin{align*}
    x + 6y + 2z - 5u - 2v &= -4 \\
    2z - 8u - v &= 3 \\
    v &= 7
\end{align*}
\]

a. (5 pts) Write the system as a matrix equation.

\[
\begin{bmatrix}
    1 & 6 & 2 & -5 & -2 \\
    0 & 0 & 2 & -8 & -1 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    u \\
    v
\end{bmatrix}
= 
\begin{bmatrix}
    -4 \\
    3 \\
    7
\end{bmatrix}
\]

b. (8 pts) Solve the system and use vector parameter form for your solution.

\[
\begin{bmatrix}
    1 & 6 & 2 & -5 & -2 \\
    0 & 0 & 2 & -8 & -1 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}
\xrightarrow{\text{RREF}}
\begin{bmatrix}
    1 & 0 & 0 & 3 & 0 & 0 \\
    0 & 0 & 1 & -4 & 1 & 5 \\
    0 & 0 & 0 & 0 & 1 & 7
\end{bmatrix}
\]

\[
\begin{align*}
x &= -6y - 3u \\
z &= 5 + 4u \\
v &= 7
\end{align*}
\]

\[
\begin{bmatrix}
    x \\
z \\
v
\end{bmatrix}
= 
\begin{bmatrix}
    -6y \\
5 + 4u \\
7
\end{bmatrix}
+ 
\begin{bmatrix}
    3u \\
0 \\
0
\end{bmatrix}
\]

for all \(u, v \in \mathbb{R}\)

2. (10 points) Consider the vectors \(\vec{u}, \vec{v}, \vec{w}\) as labelled on the graph.

a. (5 pts) Sketch and label \(2\vec{u}\) and \(\vec{v} + \vec{w}\).

b. (5 pts) Describe, geometrically and in words, the space \(\text{Span}\{\vec{u}, \vec{v}, \vec{w}\}\).

\(\text{Span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3\), plane

all linear combinations of \(\vec{u}, \vec{v}, \vec{w}\)

\(\text{Span}\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^2\), plane

all linear combinations of \(\vec{u}, \vec{v}, \vec{w}\)
3. (20 points)
   a. (2 pts) Give the definition of what it means for the set of vectors $S = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p\}$ to be linearly independent. (Don’t tell me the method you would use to SHOW they are linearly independent. I am looking for the definition.)

   If $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_p\vec{x}_p = \vec{0}$ has only the trivial solution, $c_1 = c_2 = c_3 = \ldots = c_p = 0$ then $S$ is a set of linearly independent vectors.

   Now, suppose $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$ be the columns vectors of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & -1 & 3 \\ 1 & -2 & 1 & 1 \\ -2 & -1 & 1 & -3 \end{bmatrix}$

   b. (5 pts) Find all solutions of $A\vec{x} = \vec{0}$. Hint: $RREF(A) = \begin{bmatrix} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

   Let $\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$

   $c_1 = -4/3$

   $c_2 = 0$

   $c_3 = 1/3 c_4$

   or $c_4$ is free

   c. (4 pts) Express $\vec{0}$ as a linear combination of the columns of $A$ where the scalar associated with $\vec{x}_3$ is one or explain why this is impossible.

   From b, let $c_4 = 3$ then $\vec{x} = \begin{bmatrix} -9 \\ 0 \\ 3 \end{bmatrix}$ solves $A\vec{x} = \vec{0}$.

   So, $-4\vec{x}_1 + \vec{x}_3 + 3\vec{x}_4 = \vec{0}$

   d. (5 pts) Express $\vec{x}_3$ as a linear combination of the other columns or explain why this is impossible.

   From c,

   $\vec{x}_3 = 4\vec{x}_1 - 3\vec{x}_4$

   e. (3 pts) Given some vector $\vec{b} \in \mathbb{R}^4$, can you always solve $A\vec{x} = \vec{b}$? Explain.

   By Thm 4, since $A$ doesn’t have a pivot in every row we cannot always solve $A\vec{x} = \vec{b}$ for any $\vec{b}$. 
4. (12 points) One way to see if a linear transformation \( T(\vec{x}) = A\vec{x} \) is one-to-one is to check which vectors \( \vec{x} \) in the domain are mapped to the zero vector, \( \vec{0} \) (ie. Find all \( \vec{x} \) such that \( T(\vec{x}) = A\vec{x} = \vec{0} \)). We know \( \vec{x} = \vec{0} \) will work, but there may be more in which case \( T \) is not one-to-one.

a. (6 pts) Find an example of a non-trivial (not the zero matrix) linear transformation that maps a non-zero vector to the zero vector. Give a matrix, \( A \), and a vector, \( \vec{x} \), and show how \( A \) maps your chosen \( \vec{x} \) to the zero vector.

\[
A = \begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}, \quad \vec{x} = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

Then \( A\vec{x} = \begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

b. (6 pts) A linear transformation \( T \) is one-to-one if and only if the ONLY vector that \( T \) maps to \( \vec{0} \) is \( \vec{x} = \vec{0} \) (ie. only the trivial solution). Explain why if \( T(\vec{x}) = \vec{0} \) has a unique solution, \( \vec{x} = \vec{0} \), then \( T \) is one-to-one.

If \( T(\vec{x}) = \vec{0} \) then \( A\vec{x} = \vec{0} \). If \( \vec{x} = \vec{0} \) is the only solution then \( A \) has a pivot in every column and no free variables.

However, if \( A \) has a pivot in every column then \( T(\vec{x}) = A\vec{x} \) is one-to-one.

5. (15 points) \( T \) is a linear transformation. \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \).

a. (8 pts) Can \( T \) be one-to-one? Can \( T \) be onto? Explain.

\[
T(\vec{x}) = A\vec{x} \quad A = \begin{bmatrix}
2 & x & y \\
3 & x & y
\end{bmatrix}
\]

\( T \) can be one-to-one if there is a pivot in every column of \( A \).

\( T \) can never be onto because you will never have a pivot in every row.

b. (7 pts) \( T(e_1) = \begin{bmatrix} a \\ b \\ c \\
\end{bmatrix} \) and \( T(\begin{bmatrix} -1 \\ 1 \\
\end{bmatrix}) = \begin{bmatrix} d \\ e \\ f \\
\end{bmatrix} \). Find \( T(\begin{bmatrix} 3 \\ 2 \\
\end{bmatrix}) \).

\[
T\left( \begin{bmatrix} 1 \\ 0 \\
\end{bmatrix} \right) = \begin{bmatrix} a \\ b \\ c \\
\end{bmatrix} \quad T\left( \begin{bmatrix} -1 \\ 1 \\
\end{bmatrix} \right) = \begin{bmatrix} d \\ e \\ f \\
\end{bmatrix}
\]

Since \( T \) is a linear transformation, \( T(c\vec{u}+d\vec{v}) = cT(\vec{u})+dT(\vec{v}) \)

\[
\begin{bmatrix} 3 \\ 2 \\
\end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\
\end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\
\end{bmatrix}
\]

So, \( T\left( \begin{bmatrix} 3 \\ 2 \\
\end{bmatrix} \right) = 5T\left( \begin{bmatrix} 1 \\ 0 \\
\end{bmatrix} \right) + 2T\left( \begin{bmatrix} -1 \\ 1 \\
\end{bmatrix} \right) \)

\[
= \begin{bmatrix} 5a + 2d \\ 5b + 2e \\ 5c + 2f \end{bmatrix}
\]
6. (10 points) True or False. No explanation necessary.

- T Every elementary row operation is invertible.
- T When \( \vec{u} \) and \( \vec{v} \) are nonzero vectors, \( \text{Span}\{\vec{u}, \vec{v}\} \) contains the line through \( \vec{u} \) and the origin.
- F Whenever a system has free variables, the solutions set contains many solutions.
- F The columns of a \( 4 \times 2 \) matrix always span \( \mathbb{R}^2 \).
- T My favorite theorem used to be Theorem 4, but now it is the Invertible Matrix Theorem.

7. (10 points) Given \( A \) is an \( m \times n \) matrix and \( B \) is an \( r \times t \) matrix.

a. (2 pts) Explain what must be true of \( m, n, r \) and \( t \) for \( AB \) to exist.
\[
m = r, \quad m, t \text{ free}
\]

b. (2 pts) Explain what must be true of \( m, n, r \) and \( t \) for \( BA \) to exist.
\[
t = m, \quad n, r \text{ free}
\]

c. (4 pts) Explain what must be true of \( m \) and \( n \) for \( A^2 \) to exist. What about for \( A^T \) to exist?
\[
m = n \text{ for } A^2 \text{ to exist.}
\]
\[
m \text{ and } n \text{ are free for } A^T \text{ to exist.}
\]

d. (2 pts) Suppose \( T(\vec{x}) = A\vec{x} \). \( T \) maps \( \mathbb{R}^2 \to \mathbb{R}^r \). What is \( r \) and what is \( * \).?
\[
? = n, \quad * = m
\]

8. (10 points) Find all \( a, b, c \in \mathbb{R} \) \((a,b,c \neq 0)\) for which the matrix \( A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \) solves the equation \( A^2 - a + A^T = I_2 \) where \( I_2 \) is the \( 2 \times 2 \) identity matrix.
\[
\begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} - a \begin{bmatrix} a & c \\ b & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix} a^2 + bc & ab \\ ac & cb \end{bmatrix} - \begin{bmatrix} a^2 & ac \\ ab & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
\begin{bmatrix} bc & ab - ac \\ ac - ab & cb \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
\[
bc = 1
\]
\[
ab - ac = 0 \Rightarrow a(b-c) = 0 \] since \( a \neq 0 \)
\[
ac - ab = 0 \Rightarrow a(c-b) = 0 \] (b-c) = 0 \Rightarrow b = c
\]
also \( bc = 1 \)
\[
\begin{align*}
& a = 1 & \text{or } b = 1 \text{ and } c = 1 \\
& b = 1 & \text{or } a = -1 \text{ and } c = 1 \\
& a \text{ is any non-zero real number}
\end{align*}
\]

Solutions:
\[
A = \left\{ \left( \begin{array}{cc} a & 1 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ -1 & 0 \end{array} \right) \right\}, \quad a \in \mathbb{R} \neq 0
\]