Read all of the following information before starting the exam:

- Show all work, clearly and in order if you want to get full credit (matrices can be reduced into RREF with calculator without showing steps). I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).

- Circle or otherwise indicate your final answers.

- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements. Put a smiley face next to your name for one point.

- This test has 8 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!

- Good luck!
1. *(13 points) Consider the system of equations:

\[
\begin{align*}
    x + 6y + 2z - 5u - 2v &= -4 \\
    2x - 8u - v &= 3 \\
    v &= 7
\end{align*}
\]

a. *(5 pts) Write the system as a matrix equation.

b. *(8 pts) Solve the system and use vector parameter form for your solution.

2. *(10 points) Consider the vectors \( \vec{u}, \vec{v}, \vec{w} \) as labelled on the graph.
   a. *(5 pts) Sketch and label \( 2\vec{u} \) and \( \vec{v} + \vec{w} \).

   ![Diagram with vectors \( \vec{u}, \vec{v}, \vec{w} \)]

   b. *(5 pts) Describe, geometrically and in words, the space \( \text{Span}\{\vec{u}, \vec{v}, \vec{w}\} \).
3. (20 points)
   a. (3 pts) Give the definition of what it means for the set of vectors \( S = \{ \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n \} \) to be linearly independent. (Don't tell me the method you would use to SHOW they are linear independent. I am looking for the definition.)

   Now, suppose \( S = \{ \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4 \} \) be the columns vectors of \( A = \begin{bmatrix}
   1 & 1 & 1 & 1 \\
   2 & 4 & -1 & 3 \\
   1 & -2 & 1 & 1 \\
   -2 & -1 & 1 & -3 \\
\end{bmatrix} \)

   b. (5 pts) Find all solutions of \( A \vec{z} = \vec{0} \). Hint: \( RREF(A) = \begin{bmatrix}
   1 & 0 & 0 & 4/3 \\
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & -1/3 \\
   0 & 0 & 0 & 0 \\
\end{bmatrix} \)

   c. (4 pts) Express \( \vec{0} \) as a linear combination of the columns of \( A \) where the scalar associated with \( \vec{x}_3 \) is one or explain why this is impossible.

   d. (5 pts) Express \( \vec{x}_3 \) as a linear combination of the other columns or explain why this is impossible.

   e. (3 pts) Given some vector \( b \in \mathbb{R}^4 \), can you always solve \( A \vec{z} = \vec{b} \)? Explain.
4. (12 points) One way to see if a linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one is to check which vectors $\vec{x}$ in the domain are mapped to the zero vector, $\vec{0}$ (i.e. find all $\vec{x}$ such that $T(\vec{x}) = A\vec{x} = \vec{0}$.) We know $\vec{x} = \vec{0}$ will work, but there may be more in which case $T$ is not one-to-one.

a. (6 pts) Find an example of a non-trivial (not the zero matrix) linear transformation that maps a non-zero vector to the zero vector. Give a matrix, $A$, and a vector, $\vec{x}$, and show how $A$ maps your chosen $\vec{x}$ to the zero vector.

b. (6 pts) A linear transformation $T$ is one-to-one if and only if the ONLY vector that $T$ maps to $\vec{0}$ is $\vec{x} = \vec{0}$ (i.e. only the trivial solution). Explain why if $T(\vec{x}) = \vec{0}$ has a unique solution, $\vec{x} = \vec{0}$, then $T$ is one-to-one.

5. (15 points) $T$ is a linear transformation. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$.

a. (8 pts) Can $T$ be one-to-one? Can $T$ be onto? Explain.

b. (7 pts) $T(\vec{e}_1) = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $T(\begin{bmatrix} -1 \\ 1 \end{bmatrix}) = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. Find $T(\begin{bmatrix} 3 \\ 2 \end{bmatrix})$. 

6. (10 points) True or False. No explanation necessary.
   • ___ Every elementary row operation is invertible.
   • ___ When $\vec{u}$ and $\vec{v}$ are nonzero vectors, $\text{Span}\{\vec{u}, \vec{v}\}$ contains the line through $\vec{u}$ and the origin.
   • ___ Whenever a system has free variables, the solutions set contains many solutions.
   • ___ The columns of a $4 \times 2$ matrix always span $\mathbb{R}^2$.
   • ___ My favorite theorem used to be Theorem 4, but now it is the Invertible Matrix Theorem.

7. (10 points) Given $A$ is an $m \times n$ matrix and $B$ is an $r \times t$ matrix.
   
   a. (2 pts) Explain what must be true of $m, n, r$ and $t$ for $AB$ to exist.
   
   b. (2 pts) Explain what must be true of $m, n, r$ and $t$ for $BA$ to exist.
   
   c. (4 pts) Explain what must be true of $m$ and $n$ for $A^2$ to exist. What about for $A^T$ to exist?
   
   d. (2 pts) Suppose $T(\vec{x}) = A\vec{x}$. $T$ maps $\mathbb{R}^2 \rightarrow \mathbb{R}^r$. What is $\lambda$ and what is $\lambda$.

8. (10 points) Find all $a, b, c \in \mathbb{R}$ ($a, b, c \neq 0$) for which the matrix $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ solves the equation $A^2 - a \cdot A^T = I_2$ where $I_2$ is the $2 \times 2$ identity matrix.