1. Let $T: \mathbb{R}^a \to \mathbb{R}^b$ be the linear transformation defined as $T(x) = Ax$ where $A = \begin{bmatrix} 2 & 2 & 1 \\ 9 & 7 & 3 \\ 4 & 0 & -1 \\ 11 & 5 & 1 \end{bmatrix}$

1A. What are $a$ and $b$?

$$a = \boxed{3} \quad b = \boxed{4}$$

1B. Find $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

$$T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 4 \\ 7 & 10 & 5 \\ 3 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 19 \\ 3 \\ 17 \end{bmatrix}$$

1C. Let $b = \begin{bmatrix} 1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$. Find any/all conditions that $b_1$, $b_2$, $b_3$, and $b_4$ must satisfy in order for $b$ to be in the range of $T$.

The reduced row echelon form (rref) of $A$ is:

$$\begin{bmatrix} 1 & 0 & -1/4 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 3/4 & 0 & 0 & -11/20 & 1/5 \\ 0 & 0 & 0 & 1 & 0 & 3/5 & -2/5 \\ 0 & 0 & 0 & 0 & 1 & 9/5 & -7/5 \end{bmatrix}$$

the system of equations represented here will be consistent:

$$\begin{cases} 0 = b_1 + \frac{3}{5}b_3 - \frac{2}{5}b_4 \\ 0 = b_2 + \frac{8}{5}b_3 - \frac{7}{5}b_4 \end{cases}$$

1D. Verify that $s = \begin{bmatrix} 0 \\ 5/2 \\ 10 \\ 15 \end{bmatrix}$ satisfies the condition(s) in 1C.

Does $\begin{cases} 0 \overset{?}{=} 0 + \frac{3}{5} \cdot 10 - \frac{2}{5} \cdot 15 = 3.2 - 2.3 = 6 - 6 = 0 \\ 0 \overset{?}{=} 5 + \frac{8}{5} \cdot 10 - \frac{7}{5} \cdot 15 = 5 + 8.2 - 7.3 = 5 + 16 - 21 = 0 \end{cases}$

1E. Find all $x$ such that $T(x) = s$.

"direct" ref- ing of $[A \mid \begin{bmatrix} s \end{bmatrix}]$ produces $\begin{bmatrix} 1 & 0 & -1/4 & \frac{5}{2} \\ 0 & 1 & 3/4 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, saying $T(x) = s$ $\iff$

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{4}x_3 \\ x_2 = -\frac{5}{2} - \frac{3}{4}x_3 \\ x_3 \text{ is free} \end{cases}$$

1C can be used since its 1st row says $x_1 - \frac{1}{4}x_2 = x_4 b_2 - \frac{1}{4} = \frac{5}{2}$, 2nd row gives $x_2 + \frac{3}{4}x_3 = -\frac{11}{20} b_3 + \frac{1}{5} b_4$

$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{4}x_3 \\ x_2 = -\frac{5}{2} - \frac{3}{4}x_3 \\ x_3 \text{ is free} \end{cases}$$

Note that 1C can be used since its 1st row says $x_1 - \frac{1}{4}x_2 = x_4 b_2 - \frac{1}{4} = \frac{5}{2}$, 2nd row gives $x_2 + \frac{3}{4}x_3 = -\frac{11}{20} b_3 + \frac{1}{5} b_4$

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$$\begin{cases} x_1 = \frac{5}{2} + \frac{1}{4}x_3 \\ x_2 = -\frac{5}{2} - \frac{3}{4}x_3 \\ x_3 \text{ is free} \end{cases}$$
NOTE: This is problem 1 continued!

1f. Suppose \( \mathbf{d} = \begin{bmatrix} -3 \\ 2 \\ d_3 \\ d_4 \end{bmatrix} \). Use the conditions in (1C) to find all values of \( d_3 \) and \( d_4 \) for which \( \mathbf{d} \) is in the range of \( T \). (Note you will be setting up a little linear system, and you should use our linear algebra techniques to solve it).

we have
\[
\begin{cases}
0 = b_1 + \frac{7}{5} b_2 - \frac{7}{5} b_4 \\
0 = b_2 + \frac{7}{5} b_3 - \frac{7}{5} b_4
\end{cases}
\]
from 1C and \( \mathbf{d} \) must satisfy these eqns, that is, we require
\[
\begin{cases}
0 = -3 + \frac{7}{5} d_3 - \frac{7}{5} d_4 \\
0 = 2 + \frac{7}{5} d_3 - \frac{7}{5} d_4
\end{cases}
\]
The augmented matrix representing this system is
\[
\begin{bmatrix}
\frac{7}{5} & -\frac{7}{5} & 3 \\
\frac{7}{5} & -\frac{7}{5} & -2
\end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 25 \\ 0 & 1 & 30 \end{bmatrix} \Rightarrow \begin{cases} d_3 = 25 \\ d_4 = 30 \end{cases}
\]

1G. Is \( T \) onto \( \mathbb{R}^b \)? Explain why or why not.

\( \text{No} \) because there are \( b \)'s in \( \mathbb{R}^b \) for which \( T(\mathbf{x}) = \mathbf{b} \) have no solution \( \mathbf{x} \).

Such solutions exist \( \iff \) the entries of \( b \) satisfy the two eqns in (c)
and clearly, many \( b \)'s don't satisfy them...

1H. Is \( T \) one-to-one? Explain why or why not.

\( \text{No; the presence of a free variable in } A \mathbf{x} = \mathbf{b} \text{ means } 
\text{that } T(\mathbf{x}) = \mathbf{b} \text{ can have more than one solution } \mathbf{x}; 
\text{indeed it will have \( \infty \)-many if } \mathbf{b} \text{ satisfies the conditions in (c).} \)
2. Let \( c_1, c_2, \ldots, c_6 \) be the column vectors of \( Q = \begin{bmatrix} 6 & 7 & 10 & 1 & 8 & 17 \\ 4 & 6 & 12 & 2 & 5 & 18 \\ 3 & 3 & 3 & 1 & 4 & 11 \\ 2 & 1 & -2 & 0 & 3 & 4 \end{bmatrix} \) and let \( b = \begin{bmatrix} 79 \\ 66 \\ 36 \\ 15 \end{bmatrix} \).

2A. Use your calculator to find \( \text{rref}([Q|b]) \) and copy the result here:

\[
\begin{bmatrix}
1 & 0 & -3 & 0 & 0 & 2 & -3 \\
0 & 1 & 4 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 1 & 0 & 5 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 4
\end{bmatrix}
\]

2B. Find all solutions of \( Qx = b \) and express them in the parametric form \( x = p + v_h \) where \( p \) is a particular solution of \( Qx = b \) and \( v_h \) gives all solutions of the corresponding homogeneous equation. Circle and label the two parts of your answer.

First we find the solutions \( \hat{x} \):

From 2A,

\[
\begin{align*}
x_1 &= -3 + 3x_3 - 2x_6 \\
x_2 &= 9 - 4x_3 \\
x_3 & \text{ is free} \\
x_4 &= 2 - 5x_6 \\
x_5 &= 4 \\
x_6 & \text{ is free}
\end{align*}
\]

So \( \hat{x} = \begin{bmatrix} -3 \\ 9 \\ 2 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -2 \\ 0 \\ -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \)

2C. Show that \( c_2 \) can be expressed as a linear combination of the other five column vectors; give an explicit LC.

We start with a solution of \( \hat{Q}\hat{x} = \hat{0} \) in which the weight of \( \hat{c}_2 \) is non-zero. Take \( x_3 = 1 \) and \( x_6 = 0 \) in \( \hat{v}_h \) to get

\[
\begin{align*}
x_1 &= 3 \\
x_2 &= -1 \\
x_3 &= 1 \\
x_4 &= 0 \\
x_5 &= 0 \\
x_6 &= 0
\end{align*}
\]

Thus

\[
3\hat{c}_1 - 4\hat{c}_2 + 1\hat{c}_3 + 0\hat{c}_4 + 0\hat{c}_5 + 0\hat{c}_6 = \hat{0}
\]

and there are so many ways depending on one choice of \( x_2 \) and \( x_6 \).

NOTE WELL: do NOT include \( \hat{p} \) in finding your weights, otherwise you'll be starting with \( \hat{Q}\hat{x} = \begin{bmatrix} 79 \\ 66 \\ 36 \\ 15 \end{bmatrix} \)

and your final LC will not only involve \( \hat{c}_1, \hat{c}_3, \hat{c}_5, \hat{c}_6 \), but \( \hat{c}_2 \) as well!!

2D. At least one of the column vectors is not a linear combination of the others. Find such a vector and explain why it can't be so expressed.

The vector \( \hat{v}_h \) shows that any set of \( \hat{Q}\hat{x} = \hat{0} \) forces the weight of \( \hat{c}_5 \) to be 0.

Now, if \( \hat{c}_5 \) is a LC of the other vectors, then

\[
\alpha_1\hat{c}_1 + \alpha_2\hat{c}_2 + \alpha_3\hat{c}_3 + \alpha_4\hat{c}_4 + \alpha_5\hat{c}_5 + (1)\hat{c}_6 = \hat{0}
\]

for some weights \( \alpha_1, \ldots, \alpha_6 \); this yields

\[
\alpha_1\hat{c}_1 + \alpha_2\hat{c}_2 + \alpha_3\hat{c}_3 + \alpha_4\hat{c}_4 + \alpha_5\hat{c}_5 + (1)\hat{c}_6 = \hat{0},
\]

an solution of \( \hat{Q}\hat{x} = \hat{0} \) in which the weight of \( \hat{c}_5 \) is \( \text{NOT} \) 0, a contradiction. So \( \hat{c}_5 \) is no such LC.

2E. Is the span of the set of column vectors of \( Q \) all of \( \mathbb{R}^4 \)? Explain your answer.

Yes, the ref of \( Q \) shows that the system represented by \( [Q|b] \) is consistent no matter what \( b \) is (since all rows are pivot rows, the dreaded [000000] can't occur), thus every \( \mathbb{R}^4 \) is some LC of the columns of \( Q \).

2F. Is the set \( \{c_1, c_2, \ldots, c_6\} \) linearly independent? Explain why or why not.

No, for example, \( Q\hat{x} = \hat{0} \) has more than just the trivial soln \( \hat{x} = (0,0,0,0,0,0) \) since there are free variables. Also, 20 shows at least one col vector is a LC of the others; this is impossible for \( \mathbb{R}^4 \) sets.
3. A model for an economy uses 4 sectors A, B, C, D. Sector B consumes 10% of the output of A, twice that much of its own output, and 2/5 of D's output. Sector C uses 1/10 of its own output and the remainder is consumed in equal portions by the other three sectors. Sector A consumes half of D's output and vice versa. D also uses 1/2 of B's output but A and D consume none of their own output. Sector C consumes as much of B's output as B itself does.

3A. Remembering that the entries of each column sum to one, what is the exchange table for this economy?

\[
\begin{array}{cccc}
\rightarrow A & \rightarrow B & \rightarrow C & \rightarrow D \\
0 & 1/10 & 3/10 & 2/5 \\
1/10 & 1/5 & 3/10 & 2/5 \\
2/5 & 1/5 & 1/10 & 0 \\
1/2 & 1/2 & 2/10 & 0
\end{array}
\]

3B. Suppose sector D has an equilibrium price of $179 billion. What are the other three equilibrium prices \( P_A, P_B \) and \( P_C \)? Label your answers.

The equations we need to solve are:

\[
\begin{align*}
0P_A + \frac{1}{10}P_B + \frac{3}{10}P_C + \frac{1}{2}P_D &= P_A \\
\frac{1}{10}P_A + \frac{1}{5}P_B + \frac{3}{10}P_C + \frac{2}{5}P_D &= P_B \\
\frac{2}{5}P_A + \frac{1}{5}P_B + \frac{1}{10}P_C + \frac{1}{10}P_D &= P_C \\
\frac{1}{2}P_A + \frac{1}{2}P_B + \frac{3}{10}P_C + 0P_D &= P_D
\end{align*}
\]

After substituting \( P_A, P_B, P_C, \) and \( P_D \) from the right side of these four equations, respectively, the augmented matrix becomes:

\[
\begin{bmatrix}
-1 & \frac{1}{10} & \frac{3}{10} & \frac{1}{2} & 0 \\
\frac{1}{10} & -\frac{4}{5} & \frac{3}{10} & \frac{3}{5} & 0 \\
\frac{2}{5} & \frac{1}{5} & -\frac{9}{10} & \frac{1}{10} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{3}{10} & -1 & 0
\end{bmatrix}
\begin{bmatrix}
P_A \\ P_B \\ P_C \\ P_D
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -139/179 \\
0 & 1 & -150/179 \\
0 & 0 & 1 & -115/179 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
P_A \\ P_B \\ P_C \\ P_D
\end{bmatrix}
\]

Given that \( P_D = 179 \), we find \( P_A = 139 \), \( P_B = 150 \), \( P_C = 115 \) billion $.
4. Suppose $T: \mathbb{R}^a \to \mathbb{R}^b$ is a transformation. Give the definitions of each of the following:

4a. $T$ is a linear transformation.

We say $T: \mathbb{R}^a \to \mathbb{R}^b$ is a linear transformation if:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^a$
2. $T(s\mathbf{u}) = sT(\mathbf{u})$ for any $\mathbf{u} \in \mathbb{R}^a$ and $s \in \mathbb{R}$

4b. $T$ is onto $\mathbb{R}^b$.

We say $T$ is onto $\mathbb{R}^b$ if for every $\mathbf{b} \in \mathbb{R}^b$, there is at least one $\mathbf{a} \in \mathbb{R}^a$ for which $T(\mathbf{a}) = \mathbf{b}$.

4c. Suppose $T: \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T \left( \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right) = \left[ \begin{array}{c} x_2x_3 + x_1^2 \\ 0 \\ 3x_1 + 2x_2 + x_3 \\ 2x_2 + 7 \end{array} \right]$. Show by example that $T$ is not a linear transformation and that it actually fails both parts of the definition in (4a); use all-different numbers as entries in any vectors you use.

Let's try $\mathbf{u} = \left[ \begin{array}{c} 1 \\ 3 \\ \frac{1}{2} \end{array} \right]$ and $\mathbf{v} = \left[ \begin{array}{c} \frac{4}{5} \\ 6 \\ 3 \end{array} \right]$ for example.

1. $T(\mathbf{u} + \mathbf{v}) = T \left( \begin{array}{c} \frac{9}{2} \\ \frac{21}{5} \\ 1 \end{array} \right) = \left[ \begin{array}{c} \frac{63}{5} + 25 \\ 0 \\ 15 + 14 + 9 \\ 14 + 2 \end{array} \right] = \left[ \begin{array}{c} 88 \\ 0 \\ 38 \\ 21 \end{array} \right]$

Whereas $T(\mathbf{u}) + T(\mathbf{v}) = T \left( \begin{array}{c} 1 \\ 3 \\ \frac{1}{2} \end{array} \right) + T \left( \begin{array}{c} \frac{4}{5} \\ 6 \\ 3 \end{array} \right) = \left[ \begin{array}{c} 7 \\ 0 \\ 10 \\ 11 \end{array} \right] + \left[ \begin{array}{c} 46 \\ 0 \\ 28 \\ 17 \end{array} \right] = \left[ \begin{array}{c} 53 \\ 0 \\ 38 \\ 28 \end{array} \right]$

Since $\left[ \begin{array}{c} 88 \\ 0 \\ 38 \\ 21 \end{array} \right] \neq \left[ \begin{array}{c} 53 \\ 0 \\ 38 \\ 28 \end{array} \right]$ we've shown this $T$ fails part 1 of the definition in (4a), above.

2. Let $\mathbf{u} = \left[ \begin{array}{c} 1 \\ 3 \\ \frac{1}{2} \end{array} \right]$ and $s = 2$.

$T(2\mathbf{u}) = T \left( \begin{array}{c} 2 \\ 6 \\ 1 \end{array} \right) = T \left( \begin{array}{c} 2 \\ 4 \\ 6 \end{array} \right) = \left[ \begin{array}{c} 24 + 4 \\ 0 \\ 6 + 8 + 4 \\ 8 + 7 \end{array} \right] = \left[ \begin{array}{c} 28 \\ 0 \\ 20 \\ 15 \end{array} \right]$

Whereas $2T(\mathbf{u}) = 2 \left[ \begin{array}{c} \frac{1}{2} \\ 3 \\ \frac{1}{2} \end{array} \right]$

$= 2 \left[ \begin{array}{c} \frac{3}{2} \\ 6 \\ 1 \end{array} \right] = \left[ \begin{array}{c} 14 \\ 0 \\ 2 \end{array} \right]$; since $\left[ \begin{array}{c} 28 \\ 0 \\ 20 \\ 15 \end{array} \right] \neq \left[ \begin{array}{c} 14 \\ 0 \\ 2 \end{array} \right]$, this $T$ also fails part 2 of the definition.
5. Suppose the solutions of a matrix equation $Ax = b$ are written in the form $p + v_h$, where $p$ is a particular solution of $Ax = b$ and $v_h$ gives all solutions of the corresponding homogeneous equation.

Suppose $b = \begin{bmatrix} 2 \\ -13 \\ 0 \\ 4 \end{bmatrix}$, $p = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix}$ and $v_h = x_2 \begin{bmatrix} 8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$, where $x_2$ and $x_5$ are free.

5A. Although we don't know what the original $A$ is, it is possible to say what the RREF of the augmented matrix $[A|b]$ is. Give it:

The solutions represented by $p + v_h$ tell us:
- $x_1 = 3 + 8x_2 - 7x_5$
- $x_2$ is free
- $x_3 = 6 + 2x_5$
- $x_4 = -5$
- $x_5$ is free

Since there are 5 variables, or weights, there must be 5 columns in $A$. Since $b \in \mathbb{R}^4$, all vectors in $A$ must have 4 entries, so $A$ is a $4 \times 5$ matrix.

These equations tell us 3 rows of $\text{rref}(A|b)$ must be:
- $\begin{bmatrix} 1 & -8 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

and we need to add a row of 0's since $A$ is $4 \times 5$:

Finally:
- $\begin{bmatrix} 1 & -8 & 0 & 0 & 7 & | & 3 \\ 0 & 0 & 1 & 0 & -2 & | & 6 \\ 0 & 0 & 0 & 1 & 0 & | & -5 \\
0 & 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$

5B. Label the columns of (the unseen) $A$ as $c_1$, $c_2$, ..., $c_k$. Is the set $S = \{c_1, c_2, ..., c_k\}$ linearly independent? Explain in terms of the definition of linear independence.

The columns of $A$ form a L.I. set $\iff$ the only solution to $x_1 c_1 + ... + x_5 c_5 = 0$ is the trivial solution $x_1 = ... = x_5 = 0$. But since $\text{rref}(A)$ shows there are free variables, this equation has infinitely many solutions, not just the trivial one.