NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.

Advice: DON’T spend too much time on a single problem.

<table>
<thead>
<tr>
<th>Problems</th>
<th>Maximum Score</th>
<th>Your Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. (20 pts.) (i) Is \( y = 2x^5 \) a solution of the Initial Value Problem
\[
y' = \frac{10y}{x} \quad \text{with } y(1) = 2?
\]
Justify your answer.

For \( y = 2x^5 \), it is true that \( y(1) = 2(1)^5 = 2 \). However, the D.E.
\[
y' = \frac{10y}{x}
\]
does not hold because \( y' = 2(5)x^4 = 10x^4 \) while \( \frac{10y}{x} = \frac{10(2x^5)}{x} = 20x^4 \).

Thus, \( y = 2x^5 \) is NOT a solution of this IVP.

(ii) Suppose \( f \) is a function such that \( f(2) = 2 \) and \( f'(2) = 1 \). Let \( h(x) = \frac{1}{2}f(x) + x \). Find an equation of the line tangent to the graph of \( h \) at the point \( x = 2 \).

To find the tangent line to \( h \) at \( x = 2 \), we need to find the slope which is equal to \( h'(2) \). Since \( h(x) = \frac{1}{2}f(x) + x \), it follows that \( h'(x) = \frac{1}{2}f'(x) + 1 \) so
\[
h'(2) = \frac{1}{2}f'(2) + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2}.
\]
The tangent line contains the point \( (2, h(2)) = (2, 3) \). Thus the equation of the desired tangent line is
\[
\frac{y - 3}{x - 2} = \frac{3}{2} \quad \text{or} \quad y = \frac{3}{2}x.
\]
2. (15 pts.) The graph of the derivative function $g'$ is shown below.

(i) For what values of $x$ is the function $g$ increasing?

$g$ is increasing when $g' > 0$, i.e., for $-2 < x < 1$.

(ii) For what values of $x$ is the function $g$ decreasing?

$g$ is decreasing when $g' < 0$, i.e., for $1 < x < 3$.

(iii) For what values of $x$ is the function $g$ concave up?

$g$ is concave up when $g'$ is increasing, i.e., $-1 < x < 0, 2 < x < 3$.

(iv) For what values of $x$ is the function $g$ concave down?

$g$ is concave down when $g'$ is decreasing, i.e., $-2 < x < -1, 0 < x < 2$.

(v) Find all the stationary points (if any) of $g$.

$g$ has a stationary point $x$ if $g'(x) = 0$ or the slope is zero. Thus, $g$ has a stationary point at $x = -1, 1, 3$. 
3. (16 pts.) Suppose the following is the graph of the derivative \( f' \) and \( f(0) = 2 \).

\[ f' \]

(i) Find all values of \( x \) (if any) at which \( f(x) \) is a local maximum.

For \( x \) to be a local maximum, \( f' \) must change sign from positive to negative, i.e., the graph of \( f' \) must cross the \( x \)-axis (from above the axis to below the axis). Since the graph of \( f' \) stays above the \( x \)-axis, \( f \) does NOT have any local maximum.

(ii) Find all values of \( x \) (if any) at which \( f \) has an inflection point.

The function \( f \) has an inflection point at a point \( x \) if \( f''(x) = 0 \) and \( f'' \) changes sign. Here, \( f'' \) is the slope of \( f' \). It follows from the graph of \( f' \) that \( f \) has an inflection point at \( x = 1 \) and at \( x = 3 \).

(iii) Find all values of \( x \) in the interval \( 0 \leq x \leq 4 \) such that \( f(x) \) is the minimum.

Since \( f' \geq 0 \) for \( 0 \leq x \leq 4 \) so \( f \) never decreases. This means that the minimum value of \( f \) must occur at the initial point \( x = 0 \). Thus, \( f \) attains its minimum at \( x = 0 \).

(iv) Find the limit \( \lim_{x \to 0} \frac{f(x) - 2}{x} \).

Since \( f(0) = 2 \), this limit is simply

\[ \lim_{x \to 0} \frac{f(x) - 2}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0). \]

From the graph of \( f' \), we know that \( f'(0) = 1 \). Thus, \( \lim_{x \to 0} \frac{f(x) - 2}{x} = 1 \).
4. (15 pts.) (i) Let
\[ f(x) = \frac{7}{\sqrt[3]{x}} + 5\sqrt[2]{x^7} - \frac{1}{x}. \]
Find the exact value of \( f'(1) \). Show your work.

Since \( f(x) = 7x^{-1/3} + 5x^{7/2} - x^{-1} \), it follows that
\[
f'(x) = 7 \cdot \left(-\frac{1}{3}\right) x^{-4/3} + 5 \cdot \left(\frac{7}{2}\right) x^{5/2} - (-1)x^{-2}
\]
and so
\[
f'(1) = 7 \cdot \left(-\frac{1}{3}\right) (1) + 5 \cdot \left(\frac{7}{2}\right) (1) - (-1)(1)
\]
\[
= -\frac{7}{3} + \frac{35}{2} + 1 = \frac{97}{6}.
\]

(ii) Use the limit definition of derivative to find the exact value of \( g'(1) \) for the function \( g(x) = 2x^2 + x \).

\[
g'(1) = \lim_{x \to 1} \frac{g(x) - g(1)}{x - 1}
\]
\[
= \lim_{x \to 1} \frac{2x^2 + x - (2(1)^2 + (1))}{x - 1}
\]
\[
= \lim_{x \to 1} \frac{2x^2 + x - 3}{x - 1}
\]
\[
= \lim_{x \to 1} \frac{(2x + 3)(x - 1)}{x - 1}
\]
\[
= \lim_{x \to 1} 2x + 3
\]
\[
= 5.
\]
Consider the graphs of $F(x)$ and $G(x)$ shown below.

(a) Evaluate, if it exists, each of the following limits.

(i) \( \lim_{x \to 1^-} F(x) \)

(ii) \( \lim_{x \to 3} F(x) \)

\[ \lim_{x \to 3^+} F(x) = -1 = \lim_{x \to 3^-} F(x) \] so the limit exists and is equal to \(-1\).

(iii) \( \lim_{x \to 0} G(x) \)

(iv) \( \lim_{x \to 2^+} G(x) \)

(v) \( \lim_{x \to 0^+} F(x)G(x) \)

\[ \lim_{x \to 0^+} F(x)G(x) = \left( \lim_{x \to 0^+} F(x) \right) \left( \lim_{x \to 0^+} G(x) \right) = (1)(0) = 0. \]

(b) For what values of $x$ is the function $F$ not continuous?

$F$ is discontinuous at $x = -1, 0, 1$. 
6. (16 pts.) An apple is thrown vertically upward (under free fall conditions). Let \( s(t) \) be the position of the apple at time \( t \), measured from the ground. Suppose the initial position of the apple is 1 meter and its initial velocity is 6.3 meters per second. Assume the gravity is \( a(t) = -9.8 \) meters per second per second.

(i) What is the velocity of the apple at time \( t \)?

Note that \( a(t) = v'(t) \) where \( v(t) \) is the velocity of the apple at time \( t \). Finding the antiderivative of \( a(t) \), we have \( v(t) = -9.8t + C_1 \) for some constant \( C_1 \). The initial velocity \( v(0) \) is 6.3 so \( C_1 \) must be 6.3 and thus

\[
v(t) = -9.8t + 6.3.
\]

(ii) What is the position of the apple at time \( t \)?

Note that \( v(t) = s'(t) \) where \( s(t) \) is the position of the apple at time \( t \). Finding the antiderivative of \( v(t) \), we have \( s(t) = -9.8\frac{t^2}{2} + 6.3t + C_2 \) for some constant \( C_2 \). The initial position \( s(0) \) is 1 so \( C_2 \) must be 1 and thus

\[
s(t) = -4.9t^2 + 6.3t + 1.
\]

(iii) When does the apple reach its maximum height? [Hint: what is the velocity at this instance?]

The apple reaches its maximum height when \( v(t) = 0 \). From (i), \( v(t) = 0 \) implies that \( 9.8t = 6.3 \) or \( t = \frac{9}{14} \).

(iv) When does the apple hit the ground?

The apple hits the ground when the position is 0, i.e., \( s(t) = 0 \) or \( 0 = -4.9t^2 + 6.3t + 1 = (7t + 1)(-0.7t + 1) \). Since \( t > 0 \), it follows that \(-0.7t + 1 = 0 \) or \( t = \frac{10}{7} \).