Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit.

1. (10 pts) What is the domain of \( f(x) = \frac{\sqrt{x}}{(x + 1)(x - 2)} \)?

If \( x < 0 \) then \( f \) is undefined since we can’t take the square root of a negative number.

Also, \( f \) is undefined if \( x = -1 \) or \( x = 2 \) since these values make the denominator 0.

So the domain of \( f \) is \((0, 2) \cup (2, \infty)\).

2. (10 pts) Use algebraic methods to determine if the function \( f(x) = x^5 - 3x^3 + x \) is even, odd, or neither?

Plug \(-x\) into \( f \):

\[
f(-x) = (-x)^5 - 3(-x)^3 + (-x) = -x^5 + 3x^3 - x = -(x^5 - 3x^3 + x) = -f(x) \text{ for all values of } x.
\]

\( f(-x) = -f(x) \text{ for all } x \Rightarrow f \text{ is an odd function} \)

3. (10 pts) Consider a function \( f \) with second derivative \( f''(x) = (4x^4 + 6x^2)e^{x^2} \).

Is \( x = 0 \) an inflection point for \( f \)? Justify your answer.

Recall, an inflection point occurs where the concavity of \( f \) changes.

To investigate concavity we use \( f''(x) \).

Note: \( f''(0) = (4(0)^4 + 6(0)^2)e^{0} = 0 \) so \( x = 0 \) is a possible inflection point for \( f \).

\[
\begin{array}{c|c|c}
& f'' & f'''' \\
\hline
\text{concave up} & + & + \\
x = 0 & \text{concave up} & \text{concave up} \\
\end{array}
\]

\( f''(x) > 0 \) for \( x < 0 \) and \( x > 0 \), so the function \( f \) is concave up for \( x < 0 \) and \( x > 0 \).

The function \( f \) does not change concavity at \( x = 0 \).

\( x = 0 \) is NOT an inflection pt for \( f \).
4. (12 pts) Consider the graph of $f'$, NOT $f$, provided below.

(a) On what interval(s) is $f$ increasing?

$f$ is increasing when $f'$ is positive on $(a,c) \cup (c,g) \cup (e,g)$

(b) At which labeled point(s) is $f$ greatest?

Since $f$ is non-decreasing on $(a,g)$ then $f$ is greatest at $x=g$

(c) At which labeled point(s) is $f$ least?

Since $f$ is non-decreasing on $(a,g)$ then $f$ is least at $x=a$

(d) At which labeled point(s) does $f$ have a local maximum?

$f$ has many stationary points where $f'(x)=0$ but none of the stationary points lead to a local maximum since $f$ is non-decreasing on $(a,g)$.

$\Rightarrow$ (No local max)

5. (13 pts) Use the limit definition of the derivative to find $f'(0)$ when $f(x) = \frac{2}{x+1}$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{(0+h)+1} - \frac{2}{0+1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2}{h+1} - \frac{2}{1}}{h}$$

$$= \lim_{h \to 0} \frac{2 - 2(1+h)}{h(1+h)}$$

$$= \lim_{h \to 0} \frac{2 - 2h - 2}{h}$$

$$= \lim_{h \to 0} \frac{-2h}{h}$$

$$= \lim_{h \to 0} \frac{-2}{h+1}$$

$$= -2$$

$$f'(0) = -2$$
6. (20 pts) Consider \( f(x) = \begin{cases} 
1 - x^2, & \text{if } x < 0 \\
x^2 - 1, & \text{if } x \geq 0
\end{cases} \)

Determine the following values (if they exist)

(a) \( \lim_{x \to 0^-} f(x) \)

\[ = \lim_{x \to 0^-} (1-x^2) = 1 \]

(b) \( \lim_{x \to 0^+} f(x) \)

\[ = \lim_{x \to 0^+} \frac{x^2 - 1}{x-1} = \lim_{x \to 0^+} (x+1) = 1 \]

(c) \( f(0) \)

\[ \frac{0^2 - 1}{0 - 1} = \frac{-1}{-1} = 1 \]

(d) \( \lim_{x \to 2} f(x) \)

\[ = \lim_{x \to 2} \frac{x^2 - 1}{x-1} = \lim_{x \to 2} (x+1) = 3 \]

(e) \( \lim_{x \to 1^-} f(x) \)

\[ = \lim_{x \to 1^-} \frac{x^2 - 1}{x-1} = \lim_{x \to 1^-} (x+1) = 2 \]

\( \lim_{x \to 1^+} f(x) = 2 \) since left and right hand limits are equal.

(f) \( \lim_{x \to 1^+} f(x) \)

\[ = \lim_{x \to 1^+} (x+1) = 2 \]

(g) \( \lim_{x \to 1^-} f(x) = 2 \) since left and right hand limits are equal.

(h) \( f(1) \)

\[ f(1) \text{ is undefined since } \frac{1^2 - 1}{1 - 1} \text{ gives } \frac{0}{0} \]

(i) Is \( f \) continuous at \( x = 0 \)? Explain your answer using limits.

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = 1 \text{ and } f(0) = 1, \text{ so } f \text{ is continuous at } x=0 \text{ since } \lim_{x \to 0} f(x) = f(0) \]

(j) Is \( f \) continuous at \( x = 1 \)? Explain your answer using limits.

\[ \lim_{x \to 1} f(x) + f(1) \text{ so } f \text{ is not continuous at } x=1 \]

In fact, \( f \) has a hole in its graph at \( x=1 \) since \( \lim_{x \to 1^-} f(x) \) exists while \( f(1) \) is undefined.

(k) Does \( f'(0) \) exist? Explain your answer.

\( f'(0) \) does not exist. The graph of \( f \) shown to the right shows that \( f \) has a sharp corner at \( x=0 \). In particular, as \( x \to 0^+ \) the slope of \( f \) is 1, but as \( x \to 0^- \) the slope approaches 0.

(l) Does \( f'(1) \) exist? Explain your answer.

\( f'(1) \) does not exist since \( f \) is undefined at \( x=1 \).

Note: \( f'(x) = \begin{cases} 
-2x & \text{for } x < 0 \\
1 & \text{for } x \neq 0, \text{ except } x=1
\end{cases} \)
7. (15 pts) Imagine that you are on the surface of the moon throwing a rock straight up in the air. The rock is released at a height of 1.5 meters with an initial velocity of 25 m/sec. On the moon, the acceleration due to gravity is approximately 1.62 m/sec². What is the maximum height the rock will go?

Let \( t = \) time (in seconds)
\( h(t) = \) height of rock (in meters)
\( v(t) = \) velocity of rock (in meters/sec)
\( a(t) = \) acceleration of rock (in meters/second²)

**Given information**
- \( h(0) = 1.5 \)
- \( v(0) = 25 \) m/s
- \( a(t) = -1.62 \) m/s² \( \rightarrow \) acceleration is negative since the gravitational force pulls downward on the rock.

\[ a(t) = -1.62 \text{ m/s}^2 \ \rightarrow \ \text{antidifferentiate to find } v(t) \]
\[ \Rightarrow v(t) = -1.62t + C_1 \ \leftarrow \ \text{use initial velocity} \ : \ v(0) = 25 \ \text{to find } C_1 \]
\[ 25 = v(0) = -1.62(0) + C_1 \ \Rightarrow \ C_1 = 25 \]

Therefore, \( v(t) = -1.62t + 25 \) \( \leftarrow \ \text{antidifferentiate to find } h(t) \)
\[ \Rightarrow h(t) = -0.81t^2 + 25t + C_2 \ \leftarrow \ \text{use initial height} \ : \ h(0) = 1.5 \ \text{to find } C_2 \]
\[ 1.5 = h(0) = -0.81(0)^2 + 25(0) + C_2 \ \Rightarrow \ C_2 = 1.5 \]

Therefore, \( h(t) = -0.81t^2 + 25t + 1.5 \)

The rock reaches its max height when \( v(t) = 0 \)
\[ 0 = -1.62t + 25 \ \Rightarrow \ t = \frac{25}{1.62} \approx 15.4 \text{ sec} \]

We know this is a local max since \( h''(t) = a(t) < 0 \), which means \( h \)

The rock's maximum height is \( h(15.4) = -0.81(15.4)^2 + 25(15.4) + 1.5 \approx 194.4 \text{ m} \)

8. (10 pts) Find the equation of the line tangent to the graph of \( f(x) = 2\sqrt{x} + 3x \) at \( x = 1 \).

\[ f(x) = 2\sqrt{x} + 3x \]
\[ f'(x) = 2\left(\frac{1}{2}\right)x^{-1/2} + 3 = \frac{1}{\sqrt{x}} + 3 \]

slope at \( x = 1 \) : \( f'(1) = \frac{1}{\sqrt{1}} + 3 = 4 \)

if \( x = 1 \) then \( y = f(1) = 2\sqrt{1} + 3(1) = 5 \) \( \Rightarrow \) line is tangent at \( (1, 5) \)

Use point-slope formula to find equation of tangent line
\[ m = 4 \ \rightarrow \ (1, 5) \]
\[ y - y_1 = m(x - x_1) \iff y - 5 = 4(x - 1) \]
\[ \iff y - 5 = 4x - 4 \]
\[ \iff y = 4x + 1 \]