1. (20 points) Find the domain of the given function.

\[ f(x) = \sqrt{\frac{-x}{1 - x}} \cdot \frac{\sqrt{x + 3}}{\sqrt{5 - x}} \]

**Solution.** First of all, recall that whenever we have a square root, the expression inside should be non-negative, and whenever we have a fraction, the expression in the denominator should not be zero. Thus we have

\[ \frac{-x}{1 - x} \geq 0, \quad x \neq 1, \quad x + 3 \geq 0, \quad 5 - x > 0. \]

The last two inequalities can be rewritten as \( x \geq -3 \) and \( x < 5 \), therefore, whatever the solution of the first inequality is, we have to take a part of it inside \([-3, 5)\) and remove \( x = 1 \). In order to solve the first inequality, let's rewrite it as

\[ \frac{x}{x - 1} \geq 0, \]

then it's clear from the number line that its solution is \((-\infty, 0) \cup (1, 5)\). Now combining both intervals, we get \([-3, 0) \cup (1, 5)\).

2. (20 points) Let \( f(x) = 2x^5 - \frac{5}{9}x^9 \).

a) Where is \( f(x) \) increasing?
ob) Where is \( f(x) \) decreasing?
c) Where does \( f(x) \) have local extrema?
d) Where is \( f(x) \) concave up?
e) Where is \( f(x) \) concave down?
f) Where does \( f(x) \) have inflection points?

**Solution.** In order to answer the first three questions, we have to find the derivative, factor it, and determine the signs of the derivative on each interval between the roots:

\[ f'(x) = 10x^4 - 5x^8 = 5x^4 (2 - x^4) = -5x^4 (x^4 - 2) \]

\[ = -5x^4 (x^2 - \sqrt{2}) (x^2 + \sqrt{2}) = -5x^4 (x - \sqrt{2}) (x + \sqrt{2}) (x^2 + \sqrt{2}). \]

a) \((-\sqrt{2}, 0) \cup (0, \sqrt{2})\).
b) \((-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)\).
c) \( x = -\sqrt{2} \) is a local minimum, \( x = \sqrt{2} \) is a local maximum.

to answer the last three questions, we have to do the same with the second derivative:

\[ f''(x) = (10x^4 - 5x^8) = 40x^3 - 40x^7 = 40x^3 (1 - x^4) = -40x^3 (x^4 - 1) \]

\[ = -40x^3 (x^2 - 1) (x^2 + 1) = -40x^3 (x - 1) (x + 1) (x^2 + 1). \]

d) \((-\infty, -1) \cup (0, 1)\).
e) \((-1, 0) \cup (1, +\infty)\).
f) \( x = -1, x = 0, x = 1 \) are inflection points.

3. (20 points) Let \( f(x) = 17 + 30x - 9x^2 - 4x^3 \). Find the global extreme values of \( f(x) \) on \([-2, -1] \cup [0, 2]\).

**Solution.** First of all, let's find critical points of the function:

\[ f''(x) = 30 - 18x - 12x^2 = -6 (2x^2 + 3x - 5) \Rightarrow x = \frac{-3 \pm \sqrt{49}}{4} = \frac{-3 \pm 7}{4}. \]

Thus \( x = 1 \) and \( x = -\frac{5}{2} \) are the only critical points, and only \( x = 1 \) is within the interval in question, therefore, now we have to find the values of \( f(x) \) at \( x = 1 \) and the end-points, and then choose the largest and the smallest values:
\[
\begin{align*}
  f(-2) &= 17 - 60 - 36 + 32 = -47 \\
  f(0) &= 17 + 30 - 9 - 4 = 34 \\
  f(-1) &= 17 - 30 - 9 + 4 = -18 \\
  f(1) &= 17 + 30 - 9 - 4 = 34 \\
  f(2) &= 17 + 60 - 36 - 32 = 9.
\end{align*}
\]

Thus, \( f_{\text{max}} = 34 \) at \( x = 1 \) and \( f_{\text{min}} = -47 \) at \( x = -2 \).

4. (20 points) Let \( f(x) = x - \sqrt{x} \).

a) Find an equation of the tangent line to the graph of \( y = f(x) \) at \( x = 4 \).

b) Is there a point on the graph of \( y = f(x) \) such that the tangent line at this point is \( y = \frac{5}{6}x - \frac{3}{2} \)? If yes, find the point(s); if no, prove that there is no such point.

**Solution.**

a) If \( x_0 = 4 \), then \( y_0 = f(x_0) = 4 - \sqrt{4} = 2 \). The slope of the tangent line is \( f'(x) = 1 - \frac{1}{2\sqrt{x}} \) at \( x = 4 \):

\[
\begin{align*}
  f'(x) &= 1 - \frac{1}{2\sqrt{x}} \\
  m &= 1 - \frac{1}{2\sqrt{4}} = \frac{3}{4} \\
  y - y_0 &= \frac{3}{4}(x - 4) \\
  y &= \frac{3}{4}x - 3.
\end{align*}
\]

b) Let's find a point \( x_0 \) at which the slope is \( \frac{5}{6} \):

\[
\begin{align*}
  1 - \frac{1}{2\sqrt{x_0}} &= \frac{5}{6} \\
  \frac{1}{2\sqrt{x_0}} &= \frac{1}{6} \\
  2\sqrt{x_0} &= 6 \\
  x_0 &= 9.
\end{align*}
\]

Now we just have to check whether this point is on the given tangent line or not; since \( \frac{5}{6} \cdot 9 - \frac{3}{2} = \frac{15}{2} - \frac{3}{2} = \frac{12}{2} = 6 = y_0 \), we conclude that the given line is indeed the tangent line at \( x_0 = 9 \).

5. (20 points) Let \( f(x) = \begin{cases} 
3bx^2 - 2 & \text{if } x \geq 1 \\
3x^2 - ax^4 & \text{if } x < 1
\end{cases} \). Find the values of \( a \) and \( b \) such that \( f(x) \) is continuous and differentiable at \( x = 1 \).

**Solution.** The fact that \( f(x) \) is continuous means that \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \), and that \( f(x) \) is differentiable means that \( \lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) \). Thus we have to find the derivative and set up a system for \( a \) and \( b \):

\[
\begin{align*}
  f'(x) &= \begin{cases} 
2bx & \text{if } x > 1 \\
6x - 4ax^3 & \text{if } x < 1
\end{cases} \\
  a &= 3 - 3b \\
  b &= \frac{10}{3} \\
  a &= 3 - 3 \cdot \frac{10}{3} = 3 - 10 = -7.
\end{align*}
\]

6. (20 points) A particle moves along a straight line such that its acceleration is \( a(t) = 12t^2 - 16 \). The experiment shows that at \( t = 1 \) we have the following data: the velocity of the particle is 3, and it's 5 units far from the origin in the positive direction.

a) Find at what time(s) the velocity is 15.

b) Find a function that describes the position of the particle at time \( t \).

**Solution.** We know that \( a(t) = v'(t) \) and \( v(1) = 3 \), therefore,

\[
\begin{align*}
  v(t) &= \int a(t) \, dt = \int (12t^2 - 16) \, dt = 12 \int t^2 \, dt - 16 \int 1 \, dt = 4t^3 - 16t + C_1 \\
  v(1) &= 4 - 16 + C_1 = 3 \\
  v(t) &= 4t^3 - 16t + C_1 = 15 \\
  v(t) &= 4t^3 - 16t + 15 \\
  \frac{d}{dt} [v(t)] &= 12t^2 - 16 \\
  &= 12t^2 - 16t_0 + 15 = 4t^2_0 - 16t_0 - 0 \\
  &= 4t_0(t_0 - 2) = 0 \\
  \Rightarrow t_0 &= 2, t_0 = 0.
\end{align*}
\]

b) Now we have to use that if \( p(t) \) is the position of the particle at time \( t \), then \( v(t) = p'(t) \) and \( p(1) = +5 \), therefore,

\[
\begin{align*}
  p(t) &= \int v(t) \, dt = \int (12t^2 - 16t + 15) \, dt = 4 \int t^2 \, dt - 16 \int t \, dt + 15 \int 1 \, dt = t^4 - 8t^2 + 15t + C_2 \\
  p(1) &= 1 - 8 + 15 + C_2 = 5 \\
  &\Rightarrow C_2 = 3 \\
  &\Rightarrow p(t) = t^4 - 8t^2 + 15t - 3.
\end{align*}
\]