DO NOT WRITE HERE!

1

2

3

4

5

6

7

TOTAL

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!
1. Suppose that $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a transformation. Then we say $T$ is a linear transformation if $T$ satisfies what two conditions? (Note: in addition to a couple equalities, your conditions will include the words “for all” in appropriate places).

- $T(\vec{u}) + T(\vec{v}) = T(\vec{u} + \vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$.
- $c T(\vec{u}) = T(c \vec{u})$ for all scalars $c \in \mathbb{R}$ and vectors $\vec{u} \in \mathbb{R}^n$.

2. Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T \left( \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 3x_1x_2 \\ x_2 + 4x_3 \\ x_1x_2x_3 \end{bmatrix}$.

(2A) Use the vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and scalar $c = 10$ to illustrate whether $T$ does or does not satisfy the two conditions in problem (1).

- $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} 3 \end{bmatrix} + \begin{bmatrix} 12 \end{bmatrix} = \begin{bmatrix} 15 \end{bmatrix}$
- While $T(\vec{u} + \vec{v}) = T \left( \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} \right) = \begin{bmatrix} 60 \\ 24 \\ 100 \end{bmatrix}$. Since $\begin{bmatrix} 15 \\ 24 \\ 18 \end{bmatrix} \neq \begin{bmatrix} 60 \\ 24 \\ 100 \end{bmatrix}$, condition 1 is not satisfied.

- $c T(\vec{u}) = 10 \begin{bmatrix} 3 \end{bmatrix} = \begin{bmatrix} 90 \end{bmatrix}$
- While $T(c \vec{u}) = T \left( \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right) = T \left( \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix} \right) = \begin{bmatrix} 900 \\ 110 \\ 6000 \end{bmatrix}$
- Since $\begin{bmatrix} 90 \\ 110 \\ 600 \end{bmatrix} \neq \begin{bmatrix} 900 \\ 110 \\ 6000 \end{bmatrix}$, $T$ fails to satisfy the second condition.

(Or you could also show that $10 T(\vec{v}) \neq T(10 \vec{v})$)

(2B) Do your results in 2A say $T$ is not a linear transformation or do they support the conclusion that $T$ is a linear transformation?

Because $T$ doesn't satisfy either condition, $T$ is not a linear transformation.

(And in fact, because it fails even one condition, $T$ is NOT a L.T.)
3. Suppose that $T$ is a linear transformation defined by $T(x) = Ax$ where $A$ is the matrix

\[
\begin{bmatrix}
10 & 8 & 0 \\
7 & 5 & 3 \\
5 & 4 & 0 \\
3 & 6 & -18
\end{bmatrix}
\]

(3A) What are the domain and codomain, respectively, for this $T$?

The domain is ... $\mathbb{R}^3$  

The codomain is ... $\mathbb{R}^4$

(3B) Find the image under $T$ of the vector $\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

$$
T\left(\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}\right) = 3 \begin{bmatrix} 8 \\ 5 \\ 4 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 13 \\ 10 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 3 \\ -18 \end{bmatrix} = \begin{bmatrix} 30 \\ 15 \\ 6 \end{bmatrix}
$$

(3C) Determine if the vector $\mathbf{u} = \begin{bmatrix} 9 \\ 8 \\ 4 \\ -6 \end{bmatrix}$ is in the range of $T$. If it is, find in parametric vector form all $x$ for which $T(x) = \mathbf{u}$. But if $\mathbf{u}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

The vector $\mathbf{u}$ will be in the range of $T$ if $Ax = \mathbf{u}$ has a soln.

The RREF of $A|\mathbf{u}$ is $\begin{bmatrix} 1 & 0 & 4 & 2 \frac{3}{2} \\ 0 & 1 & -5 & -2 \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and in particular, the 3rd row represents the eqn $0x_1 + 0x_2 + 0x_3 = 1$. So $Ax = \mathbf{u}$ is inconsistent system, i.e. it has no soln, and thus, $\mathbf{u} \notin$ range of $T$.

(3C) Determine if the vector $\mathbf{v} = \begin{bmatrix} 8 \\ 7 \\ 4 \\ -6 \end{bmatrix}$ is in the range of $T$. If it is, find in parametric vector form all $x$ for which $T(x) = \mathbf{v}$. But if $\mathbf{v}$ is not in the range, explain why not. Show any RREF matrices used in making your conclusions.

Here RREF of $(A|\mathbf{v})$ is $\begin{bmatrix} 1 & 0 & 4 & 2 \frac{3}{2} \\ 0 & 1 & -5 & -2 \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The system $Ax = \mathbf{v}$ is consistent, and there is a free variable.

We have that $T\mathbf{x} = \mathbf{v}$ where

$$
\mathbf{x} = \begin{bmatrix} 2 \frac{3}{2} \\ -2 \frac{3}{2} \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} \text{ where } x_3 \text{ is free} \cdot
$$
4. Suppose that $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors that belong to $\mathbb{R}^j$ for some $j$. Give the correct definition of what it means to say $S$ is a linearly independent set.

The set $S$ is a linearly independent set $\iff$
the only solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}$
is the trivial solution $x_1 = x_2 = \cdots = x_k = 0$.

5. In each part below, find a set $S$ of vectors in $\mathbb{R}^3$ that satisfy the condition(s), or explain why there is no such set. (Three separate problems)

(5A) The set $S$ is linearly independent, and contains exactly two different non-zero vectors.

One example

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(5B) The set $S$ is linearly dependent, and contains exactly three different non-zero vectors.

A common example

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

An easy example:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(5C) The set $S$ has five vectors and is linearly independent.

This cannot occur. If $S = \{\vec{v}_1, \ldots, \vec{v}_5\}$ and each vector $\vec{v} \in \mathbb{R}^3$, then the corresponding matrix equation

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 \end{bmatrix} \vec{x} = \vec{0}$$

will have at least 2 free variables, since the matrix is $3 \times 5$, which means the equation $x_1\vec{v}_1 + \cdots + x_5\vec{v}_5 = \vec{0}$ will have (infinitely many) non-trivial solutions.
6. Let \( M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix} \). Let \( S = \{ c_1, c_2, \ldots, c_5 \} \) be the set of column vectors of \( M \).

Fact: the RREF of
\[
\begin{bmatrix}
1 & 1 & 3 & 7 & 4 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 16 & 7 \\
0 & 0 & 1 & -3 & -1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

(6A) Use the fact to find conditions on \( b_1, \ldots, b_4 \) which guarantee that \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \) is in the span of the set \( S \).

The system of equations represented by \( M \mathbf{x} = \mathbf{b} \) will be consistent
if and only if \( 0 = 6 \mathbf{b}_1 + 13 \mathbf{b}_2 - 12 \mathbf{b}_3 + 5 \mathbf{b}_4 \).

(6B) We know that each member of \( S \) is in the span of \( S \), and therefore the components of each member must satisfy the condition(s) you found in (6A). Indeed show that the components of \( c_4 \) satisfy your condition(s):

\[
\begin{bmatrix}
3 \\ 4 \\ 2 \\ -7 \\
\end{bmatrix}
\]

\[
\begin{align*}
due \quad 0 &= 7 + 13 \cdot 4 - 12 \cdot 2 + 5 \cdot (-7) \quad ? \\
\text{RHS} &= 7 + 52 - 24 - 35 \\
&= 59 - 59 \\
&= 0 \quad yes.
\end{align*}
\]

(6C) Use your condition(s) from 6A to find the value of \( b_3 \) for which \( b = \begin{bmatrix} 11 \\ 3 \\ b_3 \\ 2 \end{bmatrix} \) is in the span of the set \( S \).

\[
\begin{align*}
we \quad need \quad 0 &= 11 + 13 \cdot 3 - 12 \cdot b_3 + 5 \cdot 2 \\
\therefore \quad 12b_3 &= 11 + 39 + 10 = 60 \\
\frac{b_3}{12} &= \frac{60}{12} = 5
\end{align*}
\]
7. This problem uses the same \( M \) and \( S \) as the previous problem, and the information is copied here:

Let \( M = \begin{bmatrix} 1 & 1 & 3 & 7 & 4 \\ 3 & 1 & 4 & 4 & 3 \\ 5 & 2 & 10 & 2 & 4 \\ 4 & 2 & 13 & -7 & 1 \end{bmatrix} \). Let \( S = \{c_1, c_2, \ldots, c_5\} \) be the set of column vectors of \( M \).

Fact: the RREF of \( \begin{bmatrix} 1 & 1 & 3 & 7 & 4 & 1 & 0 & 0 & 0 \\ 3 & 1 & 4 & 4 & 3 & 0 & 1 & 0 & 0 \\ 5 & 2 & 10 & 2 & 4 & 0 & 0 & 1 & 0 \\ 4 & 2 & 13 & -7 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \) is \( \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 6 & -5 & 2 \\ 0 & 1 & 0 & 16 & 7 & 0 & -25 & 23 & -10 \\ 0 & 0 & 1 & -3 & -1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 13 & -12 & 5 \end{bmatrix} \).

7A) Which members of \( S \) can be expressed as linear combinations of the other vectors in \( S \)?

The RREF tells us that \( \vec{x} \) is a solution of \( \vec{M} \vec{x} = \vec{0} \) if and only if \( \begin{bmatrix} 0 \\ -16 \\ 3 \\ 1 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -10 \end{bmatrix} + x_5 \begin{bmatrix} 7 \\ 1 \end{bmatrix} \) where \( x_4 \) and \( x_5 \) are free.

This in \( x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_5 \vec{c}_5 = \vec{0} \), it's possible to choose \( x_4 \) and/or \( x_5 \) in ways that cause \( x_2, x_3, x_4 \), and \( x_5 \) to be nonzero, and thus when \( x_i \neq 0 \) the corresponding \( \vec{c}_i \) can be solved for \( \vec{c}_i \), for \( i = 2, 3, 4, 5 \).

But since \( x_i = 0 \) in any solution \( \Rightarrow \), it's impossible to write \( \vec{c}_i \) as a L.C. of the other vectors.

7B) Find a way to express \( c_4 \) as a linear combination of \( c_2, c_3, \) and \( c_5 \), without using weights of 0, or explain why this is impossible.

Let's begin by finding a solution of \( \vec{0} \) in which none of \( x_4 \), \( x_2 \), \( x_3 \), and \( x_5 \) are 0.

Hopefully choosing \( x_4 = x_5 = 1 \) will do (at least this choice makes two of the weights nonzero.)

So: let \( x_4 = x_5 = 1 \). Then
\[
\begin{align*}
x_2 &= -16x_4 - 7x_5 = -16 - 7 = -23 \\
x_3 &= 3x_4 + x_5 = 3 + 1 = 4
\end{align*}
\]

Thus \( -23 \vec{c}_2 + 4 \vec{c}_3 + \vec{c}_4 + \vec{c}_5 = \vec{0} \).

Solving for \( \vec{c}_4 \) we find \( \vec{c}_4 = 23 \vec{c}_2 - 4 \vec{c}_3 - \vec{c}_5 \).

(Of course there are \( \infty \)-many ways to answer this question.)