EXAM #1
10/01/2010

Do NOT write here:

1
2
3
4
5

NAME
Suggested solutions
Section: B C

1) BE NEAT
2) Show ALL your work in the space provided
3) Make CLEAR what your final answer is

GOOD LUCK!
1. Consider the function $f$ shown above. Next to it, sketch the graph of $A_f(x) = \int_2^x f(t) \, dt$. Make sure the slopes on your graph are correct. (The curved portion of $f$ is one-quarter of a circle of radius 1 centered at $(5,0)$).

(Before plotting $A_f(x)$, we made the table above right)
Also, we needed to determine the areas of base shapes in the graph of $f$.

2A. Use the method of substitution to find the following: $\int_0^1 \frac{e^{-2x}}{\sqrt{1 + e^{-2x}}} \, dx$. Show all your steps. Express the answer as a decimal number to four places after the decimal.

There are at least 3 substitutions that work:

1. $u = 1 + e^{-2x}$
   $du = -2e^{-2x} \, dx$
   and $x = 0 \Rightarrow u = 1 + e^{-2(0)} = 1$
   and $x = 1 \Rightarrow u = 1 + e^{-2(1)} = 1 + \frac{1}{e^2} \approx 1.1353$
   $s$ becomes $-\frac{1}{2} \int_0^{1.1353} \frac{e^{-2x}}{\sqrt{1 + e^{-2x}}} \, dx$

2. $u = e^{-2x}$
   $du = -2e^{-2x} \, dx$
   and $x = 0 \Rightarrow u = e^{-2(0)} = 1$
   and $x = 1 \Rightarrow u = e^{-2(1)} = \frac{1}{e^2}$
   $s$ becomes $\frac{1}{2} \int_1^{\frac{1}{e^2}} \frac{e^{-2x}}{\sqrt{1 + e^{-2x}}} \, dx$

3. $u = \frac{1}{\sqrt{1 + e^{-2x}}}$
   $du = \frac{e^{-2x}}{\sqrt{1 + e^{-2x}}} \, dx$
   and $x = 0 \Rightarrow u = \frac{1}{\sqrt{1 + e^{-2(0)}}} = \frac{1}{\sqrt{2}}$
   and $x = 1 \Rightarrow u = \frac{1}{\sqrt{1 + e^{-2(1)}}} = \frac{1}{\sqrt{e^2}}$
   $s$ is simply $\frac{1}{\sqrt{1.1353}}

2B. Just to be clear: What are the new limits (written to four decimal places) on the integral after the substitution is made?

See the line marked $\times$ in each sol above.

Note: In #2: Not sec, sec$^2$, cotan, arc tan, arc sin or ANY SUCH THING!
3. The region $S$ below is bounded by the graphs of $y = 1 + \sqrt{x}$, $y = \frac{4}{5}(x - 4)$ and $x = 4$.

**Hint:** One of the following problems may require two separate integrals!

3A. Suppose the region $S$ is rotated around the line $y = 10$. Set up the integral (or integrals) giving the exact volume of the resulting solid of revolution.

$$
\text{need to get } R \text{ and } r \text{ (in terms of } x) \\
\begin{align*}
R &= 10 - \left(\frac{4}{5}(x-4)\right) \\
r &= 10 - (1 + \sqrt{x})
\end{align*}
$$

$$
\pi \int_{x=4}^{x=9} \left(\frac{4}{5}(x-4)\right)^2 - (1 + \sqrt{x})^2 \, dx
$$

3B. Suppose the region $S$ is rotated around the $y$ axis. Set up the integral (or integrals) giving the exact volume of the resulting solid of revolution.

**Need to get } R \text{ and } r \text{ in terms of } y \text{. Note well: } R \text{ is always } 4 \text{ for } y \in [0, 3], \text{ and reaches the } y\text{-axis to } y = 1 + \sqrt{x} \text{ for } y \in [3, 4].

**TOTAL VOLUME IS THE SUM OF THESE:**

$$
\begin{align*}
\pi \int_{y=3}^{y=4} &\left(\frac{4}{5}y+4\right)^2 - (y-1)^2 \, dy \\
\pi \int_{y=0}^{y=3} &\left(\frac{3}{4}y+4\right)^2 - 4^2 \, dy
\end{align*}
$$
4. Suppose a trough shaped like the one to the right is filled with water to the one-foot mark. The trough measures 8 feet long, and cross sections are bounded by the graphs of \( y = x^2 \) and \( y = 4 \). Water weighs 62.4 pounds per cubic foot.

4A. In terms of \( y \), what is the (approximate) volume of a thin sheet of water at a height of \( y_i \) feet up from the bottom of the tank and “thickness” \( \Delta y \)?

The volume of this sheet is \( (2x)(8)\Delta y \). Now, at height \( y_i \), the value of \( x \) is \( \sqrt{y_i} \), so

\[
\text{the sheet's volume is } (2\sqrt{y_i} \times 8)\Delta y
\]

(alternatively: \( \left[ -\sqrt{y_i}, \sqrt{y_i} \right] \)

4B. In terms of \( y \), what is the distance that sheet of water has to travel if the water is to be pumped to a point 15 feet above the top of the trough?

4C. What integral represents how much work is done against gravity in pumping out all the water to a point 15 feet above the top of the trough? (Just set it up; no need to evaluate it).

\[
\int_{0}^{1} \left( 2\sqrt{y} \times 8 \right) (62.4) (19 - y) \, dy
\]

only 1 foot of water in the trough.

DISTANCE

represent FORCE

represent \( dy \)
5A. Let \( I \) be the exact value of \( \int_{a}^{b} f(x) \, dx \) where \( f \) is some continuous function on \([a, b]\). "Theorem 3" (from section 6.2) says that the error committed by \( \text{MID}(n) \) in approximating \( I \), that is, \( |I - \text{MID}(n)| \), is smaller than what expression? (Be sure to say what \( K_2 \) means in your answer).

\[
|I - \text{MID}(n)| \leq \frac{K_2 \, (b-a)^3}{24 \, n^2} \quad \text{where} \quad |f''(x)| \leq K_2 \quad \text{for all} \quad x \in [a, b].
\]

5B. Now let \( f(x) = \frac{1}{12} x^4 - \frac{2}{3} x^3 - 2x^2 \);  then \( f'(x) = \frac{1}{3} x^3 - 2x^2 - 4x \) and \( f''(x) = x^2 - 4x - 4 \).

The exact value of \( \int_{1}^{4} f(x) \, dx \) is \(-1349/20 \approx -67.45\); you do not have to check this.

What is the smallest value of \( n \) for which theorem 3 guarantees \( |I - \text{MID}(n)| < 0.0005 \)? Show your calculations. (Use the best possible \( K_2 \)).

The graph of \( f''(x) \):

(\[ \begin{array}{c}
\text{So } K_2 = 8 \text{ is "best possible" we need } n \text{ for which } \frac{K_2 (b-a)^3}{24 \, n^2} < 0.0005.
\end{array} \]

So:

\[
\frac{8(4-1)^3}{24 \, n^2} < 0.0005
\]

Therefore:

\[
\frac{8 \cdot 3^3 \cdot 3 \cdot 2000}{8 \cdot 3} < n^2 \iff 18000 < n^2 \Rightarrow 134.16... < n
\]

Since \( n \) is an integer, \( n = 135 \) will do.

5C. For your value of \( n \) in the previous problem, what is \( \text{MID}(n) \) and how far is it from the exact value? Write both answers to six places after the decimal point.

\( \text{MID}(n) \) equals?

\[ \text{MID}(135) = -67.449568 \]

the error is?

\[ 0.000432 \]

5D. What is \( \text{TRAP}(30) \) for this integral? Show any intermediate values needed to find \( \text{TRAP}(30) \).

\[
\text{TRAP}(30) = \frac{\text{LHS}(30) + \text{RHS}(30)}{2}
\]

\[
= \frac{-64.93000083... + -70.00500083}{2}
\]

\[
= \frac{-134.9350017}{2} \approx -67.4675
\]