1. Evaluate the following integrals.

\[ \int_0^{\sqrt{\pi}} x^2 \cos(x^3) \, dx \]

(a) \[ u = x^3 \]
\[ du = 3x^2 \, dx \]
\[ \frac{1}{3} \, du = x^2 \, dx \]
\[ \sin^{-1} u \Rightarrow u = \sin^{-1}(3) = \pi \]

\[ \int_0^{\frac{\pi}{2}} x \, \cos(x^2) \, dx = \frac{1}{3} \int_0^{\pi} \cos u \, du = \frac{1}{3} \sin u \bigg|_0^{\pi} = \frac{1}{3} \left( \sin \pi - \sin 0 \right) = 0 \]

(b) \[ \int_0^{\frac{\pi}{6}} \frac{\arcsin x}{\sqrt{1 - x^2}} \, dx \]

\[ u = \arcsin x \]
\[ du = \frac{1}{\sqrt{1 - x^2}} \, dx \]
\[ \frac{1}{2} \Rightarrow u = \arcsin \left( \frac{1}{2} \right) = \frac{\pi}{6} \]

\[ \int_0^{\frac{\pi}{6}} \frac{\arcsin x}{\sqrt{1 - x^2}} \, dx = \int_0^{\frac{\pi}{6}} u \, du = \frac{u^2}{2} \bigg|_0^{\frac{\pi}{6}} = \frac{1}{2} \left( \frac{\pi^2}{36} \right) - \frac{1}{2} (0) = \frac{\pi^2}{72} \]
2. Suppose 10 foot long fuel tank on a truck has trapezoidal cross sections where the base of the trapezoid is 2 feet wide, the top is 4 feet wide, and the height is 8 feet, as shown in the figure. Assume that an engine is approximately 3 feet above the top of the fuel tank and that diesel fuel weighs approximately 55.6 pounds per cubic foot. \textit{Set up, but do not evaluate}, the integral that finds the amount of work done by the fuel pump in raising fuel to the level of the engine when the tank is half full.

\textit{Take horizontal cuts to get rectangular slabs}

\text{Volume of slab} = 2x(10)\Delta y = 20\left(\frac{y+8}{8}\right)dy \text{ ft}^3

\text{Force on slab} \propto \left(55.6 \text{ lb/ft}^3\right)\left(20\left(\frac{y+8}{8}\right)\Delta y\right) \text{ ft}^3

\text{distance} = 11 - y

\[ W = \int_0^8 55.6 \left(\frac{y+8}{8}\right)(11-y)dy \quad \text{or} \quad \int_0^8 139(y+8)(11-y)dy \]
3. Consider the initial value problem $\frac{dy}{dx} = y \ln y$ with $y(1) = e$. Use separation of variables to find the solution to the IVP.

\[
\frac{dy}{dx} = y \ln y
\]

\[
\frac{dy}{y \ln y} = dx
\]

So

\[
\ln |\ln y| = x + C \quad \text{if } y(1) = e
\]

\[
\ln |\ln e| = 1 + C
\]

\[
0 = 1 + C
\]

\[
C = -1
\]

\[
\Rightarrow \ln |\ln y| = x - 1
\]

\[
|\ln y| = e^{x-1}
\]

\[
y = e^{x-1}
\]

4. Let $I = \int_0^1 2x^3 \, dx$. How many subdivisions are required to obtain a trapezoidal sum approximation with error of at most 1/10,000? Recall that the error bound estimates for trapezoidal sums may be determined using:

\[
|I - T_n| \leq \frac{K_2 (b-a)^3}{12n^2}.
\]

\[
f(x) = 2x^3
\]

\[
f'(x) = 6x^2
\]

\[
f''(x) = 12x
\]

where $|f''(x)| \leq 12$ on $[0, 1]$ therefore $K_2 = 12$

\[
|I - T_n| \leq \frac{12 \cdot (1-0)^3}{12n^2} \leq \frac{1}{10000}
\]

\[
\frac{1}{n^2} \leq \frac{1}{10000}
\]

\[
v^2 \geq 10000
\]

\[
v \geq 100
\]
5. Consider the region bounded by the curves \( x = 3y^2 - 2 \) and \( x = y^2 \) and the line \( y = 0 \).

(a) Set up, but do not evaluate, the integral which represents the volume of the solid formed by revolving the region around the \( x \)-axis.

\[
V = \int_{-1}^{0} \pi \left( \sqrt{\frac{x+2}{3}} \right)^2 \, dx + \int_{0}^{1} \pi \left[ \left( \sqrt{\frac{x+2}{3}} \right)^2 - \left( \sqrt{y} \right)^2 \right] \, dx
\]

Interpretation:
- For \( 3y^2 - 2 = x \)
- For \( y = 1 \):
  - \( x = 1 \)
  - \( x = -\frac{4}{3} \)
- For \( y = -1 \):
  - \( x = 1 \)
  - \( x = -\frac{4}{3} \)

(b) Set up, but do not evaluate, the integral which represents the volume of the solid formed by revolving the region around the line \( x = 1 \).

\[
V = \int_{0}^{1} \pi \left[ (1 - (3y^2 - 2))^2 - (1 - y^2)^2 \right] \, dy
\]

or
\[
\pi \int_{0}^{1} \left[ (3y^2)^2 - (1 - y^2)^2 \right] \, dy
\]

\[
R = 1 - (3y^2 - 2) = 3 - 3y^2
\]

\[
r = 1 - y^2
\]
6. Consider a function \( f \) where the graph of its derivative \( f' \) and tables of values is shown below.

\[
\begin{align*}
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 0 & 0.2 & 0.4 & 0.6 & 0.8 & 1.0 \\
\hline
f'(x) & 0.54 & 0.659 & 0.862 & 1.097 & 1.342 \\
\hline
\end{array}
\end{align*}
\]

(a) Write an integral that represents the arc length of \( f \) on the interval \([0, 1]\).

\[
\int_0^1 \sqrt{1 + [f'(x)]^2} \, dx
\]

(b) Use the table of values above to approximate your integral from (a) using \( R_5 \). Round to 3 digits after the decimal. Note: you might want to expand the table above by adding a row for values of the integrand.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 0 & .2 & .4 & .6 & .8 & 1.0 \\
\hline
[f'(x)]^2 & 1.25 & 0.2916 & 0.1343 & 0.7259 & 1.2034 & 1.801 \\
\hline
\end{array}
\]

\[
g(x) = \sqrt{1 + [f'(x)]^2} = 1.118 \quad 1.1365 \quad 1.1976 \quad 1.3137 \quad 1.4844 \quad 1.6736
\]

\[
R_5 = g(.2) \Delta x + g(.4) \Delta x + g(.6) \Delta x + g(.8) \Delta x + g(1) \Delta x
\]

\[
\approx 0.2 \left[ 1.1365 + 1.1976 + 1.3137 + 1.4844 + 1.6736 \right]
\]

\[
R_5 \approx 1.361
\]

(c) Determine if \( R_5 \) is an overestimate or underestimate for the exact value of the integral in (a). Justify your answer. (Hint: carefully consider the integrand in part (a) as well as characteristics about \( f' \).)

To see if \( R_5 \) is an over- or underestimate for \( \int_0^1 \sqrt{1 + [f'(x)]^2} \, dx \), we must determine if \( g(x) = \sqrt{1 + [f'(x)]^2} \) is increasing or decreasing.

i.e., \( g'(x) > 0 \) or \( g'(x) < 0 \)

\[
g'(x) = \frac{1}{2} \left( 1 + (f'(x))^2 \right)^{-1/2} \cdot 2 f'(x) f''(x) = \frac{f'(x) f''(x)}{\sqrt{1 + [f'(x)]^2}}
\]

Note: graph of \( f' \) is increasing \( \Rightarrow f'' > 0 \)

Likewise graph of table show \( f'(x) > 0 \)

Therefore \( g'(x) = \frac{f'(x)f''(x)}{\sqrt{1 + [f'(x)]^2}} > 0 \) \( \Rightarrow \sqrt{1 + [f'(x)]^2} \) is increasing.

Therefore \( \int_0^1 g(x) \, dx < R_5 \).