Problem 1. (15 points) Shown at left is a graph of the function $f(x) = x^4 - 10x^3 + 24x^2 - 8x$.

Using the formula for $f$ and its derivative(s) when necessary, clearly demonstrate each of the following.

(a) (5 points) The function $f(x)$ is decreasing at $x = 3$.

Decreasing can be recognized as having negative first derivative. Taking the derivative of $f$, we see

$$f'(x) = 4x^3 - 30x^2 + 48x - 8.$$  

Then at $x = 3$, the first derivative has value

$$f'(3) = 4(3)^3 - 30(3)^2 + 48(3) - 8 = 108 - 270 + 144 - 8 = -26.$$  

Therefore $f'(x)$ is negative at $x = 3$, so $f(x)$ is decreasing at $x = 3$.

(b) (5 points) The point at $x = 2$ is a local maximum of $f(x)$.

First, is $x = 2$ a stationary point? It will be if $f'(2) = 0$. From part (a), we have

$$f'(2) = 4(2)^3 - 30(2)^2 + 48(2) - 8 = 32 - 120 + 96 - 8 = 0.$$  

So $x = 2$ is a stationary point of $f$. From here, we can use either the first or the second derivative test to verify that this stationary point is a local maximum.

The first derivative test requires us to test points to the left and right of $x = 2$.

$x = 1$  
$f'(1) = 14$  
$f$ incr.  

$x = 2$  
$f'(2) = 0$  
$f$ stationary  

$x = 3$  
$f'(3) = -26$  
$f$ decr.

Thus $f$ is changing from increasing to decreasing at $x = 2$, indicating a local maximum.

The second derivative test can be used instead to check the concavity of $f$ at $x = 2$:

$$f''(x) = 12x^2 - 60x + 48$$  

so  
$$f''(2) = 12(2)^2 - 60(2) + 48 = 48 - 120 + 48 = -24.$$  

Since $f''(2)$ is negative, $f$ is concave down at $x = 2$, making this stationary point a local maximum.

(c) (5 points) The point at $x = 4$ is an inflection point of $f(x)$.

A function is neither concave up nor concave down at its inflection point. Thus we expect $f''(4) = 0$. And

$$f''(x) = 12x^2 - 60x + 48$$  

so  
$$f''(4) = 12(4)^2 - 60(4) + 48 = 192 - 120 + 48 = 0.$$  

(We might also want to verify that $f$ actually changes concavity at $x = 4$. This can be done by testing $x = 1$ and $x = 3$, but is not strictly necessary.)
Problem 2. (15 points) Nowadays, most new cars can accelerate from 0 mph to 60 mph in 9 seconds or less. Your professor’s first car — A 1973 Saab — had a tiny engine with only 74 horsepower. It took 16 seconds to accelerate from 0—60 mph. Below is the result of a road test, showing the velocity of the car (in ft/sec) as a function of time. A graph and a formula are given.

(a) (5 points) Find an equation for the car’s distance function \( d(t) \), measured in feet.

Hint: Your answer should contain an unknown constant!

We’re looking for a function whose derivative is \( v(t) \). Taking the antiderivative of \( v(t) \), we obtain

\[
d(t) = 22 \left( \frac{t^{1/2+1}}{1/2+1} \right) + C = 22 \left( \frac{t^{3/2}}{3/2} \right) + C = \frac{44}{3} t^{3/2} + C.
\]

We include the +C so our antiderivative captures every possibility.

(b) (5 points) Assume that at time \( t = 0 \), the car’s distance is 0 feet. Use your answer to part (a) to determine how much distance the car covered during the 16 seconds of its test.

The first sentence implies that when \( t = 0, d = 0 \). This allows us to determine a value for \( C \):

\[
0 = d(0) = \frac{44}{3} (0)^{3/2} + C \\
0 = 0 + C \\
0 = C.
\]

Thus the distance function we’re looking for really is \( d(t) = \frac{44}{3} t^{3/2} \).

After 16 seconds, then, the car has traveled a distance of

\[
d(16) = \frac{44}{3} (16)^{3/2} = \frac{2816}{3} \approx 938.666 \ldots \text{ ft}.
\]

(c) (5 points) How quickly is the car accelerating after 2 seconds? Include units in your answer.

Acceleration being the derivative of velocity, we have the acceleration function

\[
v'(t) = 22 \frac{1}{2} t^{1/2-1} = 11 t^{-1/2} = \frac{11}{\sqrt{t}}.
\]

Thus at \( t = 2 \) seconds, the acceleration is

\[
v'(2) = \frac{11}{\sqrt{2}} \approx 7.778 \frac{\text{ft}}{\text{sec}^2}.
\]

Bonus: “One g” of acceleration is about 32 ft/sec^2. How many g’s is the car pulling after 2 seconds?

Part (c)’s answer equates to \( \frac{7.778}{32} = 0.243 \text{ g} \).