1. **Find the following.** [Substitution tip: usually let \( u \) = a function that’s “inside” another function, especially if \( du \) (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let \( u = \sqrt{x} \), so \( du = \frac{dx}{2\sqrt{x}} \) and \( 2\,du = \frac{dx}{\sqrt{x}} \).

Now we’ll change the limits.

If \( x = 1 \), then \( u = \sqrt{1} = 1 \) and if \( x = 4 \), then \( u = \sqrt{4} = 2 \).

\[
\int_{1}^{4} e^{\sqrt{x}} \cdot \frac{dx}{\sqrt{x}} = \int_{1}^{2} e^u \cdot 2\,du \\
= 2e^u \bigg|_{1}^{2} \\
= 2e^2 - 2e \quad (\approx 9.342)
\]

(b) Let \( u = \cos(5x) \), so \( du = -5\sin(5x)\,dx \) and \( -\frac{du}{5} = \sin(5x)\,dx \).

Now we’ll change the limits.

If \( x = \pi \), then \( u = \cos(5 \cdot \pi) = -1 \) and if \( x = 2 \), then \( u = \cos(5 \cdot 2\pi) = 1 \).

\[
\int_{\pi}^{2\pi} \cos^7(5x) \sin(5x)\,dx = \int_{-1}^{1} u^7 \cdot -\frac{du}{5} \\
= -\frac{1}{5} \int_{-1}^{1} u^7 \,du \\
= -\frac{1}{5} \frac{u^8}{8} \bigg|_{-1}^{1} \\
= -\frac{1}{40} (1^8 - (-1)^8) \\
= 0
\]

(c) Use \( u = x^3 \), so \( du = 3x^2\,dx \) and \( \frac{du}{3} = x^2\,dx \).

\[
\int \frac{7x^2}{1 + x^6}\,dx = 7 \int \frac{du}{1 + u^2} \\
= \frac{7}{3} \arctan u + C \\
= \frac{7}{3} \arctan(x^3) + C
\]
(d) Use \(u = 10 - x\), so \(du = -dx\) and \(dx = -du\).

\[
\int x\sqrt{10-x} \, dx = \int (10-u)\sqrt{u} (-du)
\]

Since \(u = 10 - x\), we know \(x = 10 - u\).

\[
= \int (u-10)\sqrt{u} \, du
= \int (u^{3/2} - 10u^{1/2}) \, du
= \frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2} + C
= \frac{2}{5}(10-x)^{5/2} - \frac{20}{3}(10-x)^{3/2} + C
\]

2. If \(f(x)\) is decreasing and concave up, put the following quantities in ascending order.

\[L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) \, dx\]

\[R_{100} < M_{100} < \int_a^b f(x) \, dx < T_{100} < L_{100}\]

What can you say with certainty about where \(S_{200}\) would fit into your list above?

It would be somewhere between \(M_{100}\) and \(T_{100}\) but we don’t know how it compares to \(\int_a^b f(x) \, dx\).

3. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \(\int_4^{12} f(t) \, dt\) given the data in the table below.

<table>
<thead>
<tr>
<th>(t)</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(t))</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\[L_4 = (15 + 11 + 8 + 4)(2) = 76 \quad R_4 = (11 + 8 + 4 + 3)(2) = 52 \quad T_4 = \frac{L_4 + R_4}{2} = 64\]

We cannot compute \(M_4\) because it requires the values of \(f\) at \(x = 5, 7, 9,\) and \(11\). Instead, we do \(M_2\).

\[M_2 = (11 + 4)(4) = 60\]

Now, to find \(S_4\), we need \(T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68\).

\[S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.6\]

4. Find bounds for each of the following errors if \(I = \int_2^7 \ln x \, dx\).

(a) \(|I - L_{100}| \leq \frac{K_1(b - a)^2}{2n} = \frac{\frac{1}{x}(7 - 2)^2}{2(100)} = \frac{1}{16}\)

\(K_1 = \max \{|f'(x)|\} \text{ on } [2, 7] = \max \left\{ \frac{1}{x} \right\} \text{ on } [2, 7] = \frac{1}{2} \) (occurs at \(x = 2\))

(b) \(|I - T_{100}| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{\frac{1}{x^2}(7 - 2)^3}{12(100)^2} = \frac{1}{3840}\)

\(K_2 = \max \{|f''(x)|\} \text{ on } [2, 7] = \max \left\{ \frac{1}{x^2} \right\} \text{ on } [2, 7] = \frac{1}{4} \) (occurs at \(x = 2\))
5. If $I = \int_2^7 \ln x \, dx$, how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most $1/1,000,000$?

From part (b) above, we know that $|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12n^2} = \frac{1}{48n^2}$.

Thus, we want $\frac{125}{48n^2} \leq \frac{1}{1,000,000}$.

Multiplying each side by $1,000,000n^2$ gives $\frac{125,000,000}{48} \leq n^2$.

Taking the square root of each side results in $\sqrt{\frac{125,000,000}{48}} \leq n$.

Since $\sqrt{\frac{125,000,000}{48}} = 1613.743\ldots$, we must at least 1614 subdivisions.

6. Solve the differential equation $\frac{dy}{dx} = 2xy + 6x$ if the solution passes through $(0,5)$.

$$\frac{dy}{dx} = 2xy + 6x$$
$$\frac{dy}{dx} = 2x(y + 3)$$
$$\frac{dy}{y + 3} = 2x \, dx$$

Separate the variables.

$$\int \frac{dy}{y + 3} = \int 2x \, dx$$
$$\ln |y + 3| = x^2 + C$$
$$|y + 3| = e^{x^2+C}$$
$$y + 3 = \pm e^C e^{x^2}$$
$$y = -3 + Ae^{x^2}$$

Exponentiate each side to remove the ln.

$|w| = z$ means $w = \pm z$.

Replace $\pm e^C$ with $A$.

Now we use the initial condition $y(0) = 5$ to find the value of $A$.

We have $5 = -3 + Ae^0 \Rightarrow A = 8$, so the solution is $y = -3 + 8e^{x^2}$.

7. Write integrals equal to

(a) the arc length of $y = x^2$ on the interval $[1,5]$

$$\text{arc length of } y = f(x) \text{ on } [a,b] = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx \approx 24.395$$

(b) the area bounded by $y = x^2 - 8x + 24$ and $y = 3x$
First, find where the curves intersect.

\[
x^2 - 8x + 24 = 3x
\]
\[
x^2 - 11x + 24 = 0
\]
\[
(x - 3)(x - 8) = 0
\]
\[\Rightarrow x = 3, x = 8\]

Between \(x = 3\) and \(x = 8\), \(y = 3x\) is above \(y = x^2 - 8x + 24\). (Plug in \(x = 5\) or graph to check.)

So, the area between them is

\[
\int_3^8 [3x - (x^2 - 8x + 24)] \, dx.
\]

[This equals \(125/6\).]

8. Consider the region bounded by \(y = \sqrt{x}\), \(y = 0\), and \(x = 9\). Write an integral equal to the volume generated if this region is revolved about

(a) the \(x\)-axis

volume of slice \(\approx \pi r^2 \Delta x\)
\[= \pi y^2 \Delta x\]
\[= \pi (\sqrt{x})^2 \Delta x\]
\[= \pi x \Delta x\]

total volume \(= \pi \int_0^9 x \, dx\)

(b) the line \(x = -1\)

volume of slice \(\approx \pi R^2 \Delta y - \pi r^2 \Delta y\)
\[= \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y\]
\[= \pi [100 - (1 + y^2)^2] \Delta y\]

total volume \(= \pi \int_0^3 [100 - (1 + y^2)^2] \, dy\)
9. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) the volume of the pyramid

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of $\Delta h$.

The picture shown below is a vertical cross-section through the center of the pyramid.

![Pyramid cross-section diagram]

Similar triangles: \( \frac{10}{30} = \frac{10-h}{s} \Rightarrow s = 3(10-h) \).

volume of slice \( \approx s^2 \Delta h \approx [3(10-h)]^2 \Delta h \)

total volume = \( \int_{0}^{10} [3(10-h)]^2 \, dh \)

(b) the work done in pumping all the fluid to a point 5 feet above the pyramid if the pyramid is filled to a height of 8 feet with water (which weighs 62.4 pounds per cubic foot)

We use the same sketch as in the previous part.

volume of slice \( \approx s^2 \Delta h \approx [3(10-h)]^2 \Delta h \) From above.

weight of slice \( \approx 62.4[3(10-h)]^2 \Delta h \) Weight=(density)(volume).

work to lift slice \( \approx 62.4[3(10-h)]^2 \Delta h (15-h) \) Work=(force)(distance); here, force=weight.

total work = \( 62.4 \int_{0}^{8} [3(10-h)]^2 (15-h) \, dh \)