NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.

Advice: DON’T spend too much time on a single problem.

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1. (10 pts.) (a) Evaluate the indefinite integral (be sure to show all your work)

\[ \int e^x \sin(e^x) \, dx. \]

Let \( u = e^x \) so that \( du = e^x \, dx \). It follows that

\[ \int e^x \sin(e^x) \, dx = \int \sin u \, du = - \cos u + C = - \cos(e^x) + C. \]

(10 pts.) (b) Find the exact value of the definite integral (be sure to show all your work)

\[ \int_0^2 \frac{x^2}{\sqrt{x^3 + 1}} \, dx. \]

Let \( u = x^3 + 1 \) so that \( du = 3x^2 \, dx \) or \( x^2 \, dx = \frac{1}{3} \, du \). When \( x = 0, u = 1 \) and when \( x = 2, u = 9 \). It follows that

\[ \int_0^2 \frac{x^2}{\sqrt{x^3 + 1}} \, dx = \int_1^9 \frac{1/3 \, du}{\sqrt{u}} = \frac{1}{3} \int_1^9 u^{-1/2} \, du = \frac{1}{3} \left[ \frac{2}{1/2} \right]_1^9 \frac{u^{1/2}}{1} = \frac{2}{3} (3 - 1) = \frac{4}{3}. \]
2. Consider the region bounded by the curve $y = \frac{1}{3}x^2$ and the curve $y = x(4 - x)$.

(5 pts.) Sketch a picture of the region by determining the points of intersection between the curves.

(15 pts.) Find the exact area of the region described in part (a).

The area of the region described in part (a) is given by

\[
\int_{0}^{3} x(4 - x) - \frac{1}{3}x^2 \, dx \\
= \int_{0}^{3} 4x - x^2 - \frac{x^2}{3} \, dx \\
= \int_{0}^{3} 4x - \frac{4}{3}x^2 \, dx \\
= 2x^2 - \frac{4}{9}x^3 \bigg|_{0}^{3} \\
= 18 - 12 = 6.
\]
3. (14 pts.) Consider a function \( g \) on the interval \([-2, 4]\).

\[
\begin{array}{c|cccccc}
 x & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
g(x) & 1 & 0 & -1 & -2 & 1 & 3 & 4 \\
\end{array}
\]

Find \( L_6, M_3 \) using the left-hand sum and the mid-point rule respectively for estimating the definite integral \( \int_{-2}^{4} g(x) \, dx \).

**For \( L_6 \), \( \Delta x = 1 \) so that**

\[
L_6 = [(1) + (0) + (-1) + (-2) + (1) + (3)] \cdot (1) = 2.
\]

**For \( M_3 \), \( \Delta x = 2 \) so that**

\[
M_3 = [(0) + (-2) + (3)] \cdot (2) = 2.
\]

4. (6 pts.) Set up (do not evaluate) a definite integral for the arc length of the portion of the graph of \( f(x) = e^{-x} \sin x \) between \( x = 0 \) and \( x = \frac{\pi}{2} \).

**First, the derivative** \( f'(x) = (-e^{-x}) \sin x + e^{-x} \cos x = e^{-x}(\cos x - \sin x) \). **Thus the required arc length is given by the following definite integral**

\[
\int_{0}^{\pi/2} \sqrt{1 + e^{-2x} \cos x - \sin x}^2 \, dx.
\]
5. Let \( R \) be the region bounded by the curve \( 2(x - 2) = y^2 \), the line \( y = 2 \), the line \( y = x \), and the \( x \)-axis.

(10 pts.) (a) Set up (do not evaluate) a definite integral representing the volume of the solid obtained from rotating the region \( R \) around the line \( x = 0 \), i.e., the \( y \)-axis. [Hint: sketch a picture of the region \( R \) first.]

The volume of the solid in question is given by

\[
\int_{0}^{2} \pi \left( 2 + \frac{y^2}{2} \right)^2 - \pi y^2 \ dy.
\]

(10 pts.) (b) Find the exact volume of the solid described in part (a).

\[
\int_{0}^{2} \pi \left( 2 + \frac{y^2}{2} \right)^2 - \pi y^2 \ dy = \pi \int_{0}^{2} \left( 4 + 2y^2 + \frac{y^4}{4} \right) - y^2 \ dy
\]

\[
= \pi \int_{0}^{2} 4 + y^2 + \frac{y^4}{4} \ dy
\]

\[
= \pi \left[ 4y + \frac{y^3}{3} + \frac{y^5}{20} \right]_{0}^{2}
\]

\[
= \pi \left[ 8 + \frac{8}{3} + \frac{8}{5} \right] = \frac{184\pi}{15}.
\]
6. Consider the initial value problem
\[ \frac{dy}{dx} = \frac{y + x^2y}{x^2} \]
with \( y(1) = 1 \).

(10 pts.) (a) Use Euler’s method to estimate the value \( y(3) \) (when \( x = 3 \)) using two steps with initial point (1, 1). DO THIS BY HAND and show all your steps.

Here, the step size \( \Delta x = 1 \). Thus, we have

\[
y_1 = y(0) + \frac{dy}{dx}\bigg|_{(1,1)} \cdot \Delta x = 1 + \frac{1 + 1}{1} \cdot 1 = 3.
\]

\[
y_2 = y_1 + \frac{dy}{dx}\bigg|_{(2,3)} \cdot \Delta x = 3 + \frac{3 + (2)^2(3)}{(2^2)} \cdot 1 = \frac{27}{4}.
\]

Thus, \( y(3) \approx \frac{27}{4} \).

(10 pts.) (b) Use the technique of separation of variables to solve the Initial Value Problem.

Separating the variables yields

\[
\frac{1}{y} \, dy = \frac{1 + x^2}{x^2} \, dx = x^{-2} + 1 \, dx.
\]

It follows that

\[
\int \frac{1}{y} \, dy = \int x^{-2} + 1 \, dx = -x^{-1} + x + C
\]

and thus

\[
\ln y = x - \frac{1}{x} + C.
\]

Since \( y(1) = 1 \), it follows that \( 0 = \ln 1 = 1 - \frac{1}{1} + C \) or \( C = 0 \). We conclude that

\[
\ln y = x - \frac{1}{x} \quad \text{or} \quad y = e^{x - \frac{1}{x}}.
\]