1. Do not open this exam until you are told to do so.

2. This exam has 7 pages including this cover AND IS DOUBLE SIDED. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.

3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out when you hand in the exam.

4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions.

5. Show an appropriate amount of work (including appropriate explanation). Include units in your answer where that is appropriate. Time is of course a consideration, but do not provide no work except when specified.

6. You may use any previously permitted calculator. However, you must state when you use it.

7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph that you use.

8. Turn off all cell phones and pagers, and remove all headphones and hats.

9. Remember that this is a chance to show what you’ve learned, and that the questions are just prompts.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
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<tbody>
<tr>
<td>1</td>
<td>22</td>
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<tr>
<td>2</td>
<td>18</td>
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<td>3</td>
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<tr>
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</tbody>
</table>
Some integrals you may use:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( \cos(x) )</th>
<th>( \sin(x) )</th>
<th>( e^x )</th>
<th>( \frac{1}{\sqrt{1-x^2}} )</th>
<th>( \frac{1}{1+x^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int f(x) , dx )</td>
<td>( \sin(x) + C )</td>
<td>( -\cos(x) + C )</td>
<td>( e^x + C )</td>
<td>( \sin^{-1}(x) + C )</td>
<td>( \tan^{-1}(x) + C )</td>
</tr>
</tbody>
</table>
1. [22 points] Let \( I = \int_0^5 f(x)\,dx \) where \( f(x) \) is increasing over the interval \([0, 5]\).
Indicate whether the statement must be true, cannot be true, or may be true. Briefly explain your choices, no credit for no explanation.

a. [4 points]
For any \( n \geq 1 \) it’s true that \( \text{Left}_n \geq I \)

b. [4 points]
For any \( n \geq 1 \) it’s true that \( \text{Trap}_n \geq I \)

Solution: It is the case that \( \text{Left}_n \) CANNOT be greater or equal to \( I \), because the function is increasing. We know that for increasing functions, \( \text{Left}_n < I \).
If the function is concave up, then \( \text{Trap}_n \geq I \), if the function is concave down, then \( T_n \leq I \), and since this function can be either, then it MAY be true that \( \text{Trap}_n \leq I \).

c. [7 points] Given the following table, calculate \( \text{Trap}_5 \) (the trapezoid rule with 5 intervals):

<table>
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<tr>
<th>x</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
<th>3.5</th>
<th>4</th>
<th>4.5</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>-3</td>
<td>-2</td>
<td>0</td>
<td>1.5</td>
<td>2</td>
<td>5</td>
<td>6</td>
<td>6.5</td>
<td>8</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

Solution: Using our Trapezoid formula, we need to average the Left hand and Right hand approximations:

\[
\frac{1}{2} \times 1 \times (-3 + 2 \times (0 + 2 + 6 + 8) + 11) = 20
\]

d. [7 points] Given the additional facts that \(-5 \leq f'(x) \leq 3\) and \(-9 \leq f''(x) \leq 7\) on the interval \([0, 5]\) use your previous calculation to find a range of possible values for \( I \).

Solution: Given our formula that the error is bounded by:

\[
|I - \text{Trap}_4| \leq \frac{K_2(b - a)^3}{12n}.
\]

We can then choose \( K_2 = 9 \geq |f''(x)| \) and \( b = 5 \), \( a = 0 \) and \( n = 4 \). This gives that the error is at most

\[
\frac{9 \times 125}{48} \approx 23.44.
\]

So we have that

\[
20 - 23.44 \leq I \leq 20 + 23.44.
\]
2. [18 points] Calculate two of the following integrals (9 points each)—Show ALL work and proper notation:

1. \[ \int_{0}^{\pi} \frac{3 \cos(x) e^{\sin(x)}}{\sin^3(x) + 2} \, dx \]

2. \[ \int_{1}^{2} \frac{e^{(-\frac{1}{x})}}{x^2} \, dx \]

3. \[ \int_{-10}^{10} \frac{1}{w^2 + 2w + 2} \, dw \]

Solution:

1. We see the inside function \( \sin(x) \) so we substitute \( u = \sin(x) \) and \( du = \cos(x) \, dx \). We must also change the bounds: the lower bound becomes \( \sin(0) = 0 \) and the upper bound becomes \( \sin(\pi) = 0 \). Therefore the integral equals \( \int_{0}^{0} \frac{3e^{u}}{u^3 + 2} \, du \). But because the bounds are the same number, the integral is 0.

2. We see the inside function \( -\frac{1}{x} \) and so we substitute \( u = -\frac{1}{x} \) and \( du = \frac{dx}{x^2} \). The bounds are summarily changed to 1 and \( \frac{1}{2} \). This gives the new integral \( \int_{1}^{\frac{1}{2}} e^u \, du = e^{1/2} - e \).

3. Finally, we need to complete the square in the denominator so we get \( w^2 + 2w + 2 = (w + 1)^2 + 1 \). We then substitute our inside function \( w + 1 = u \) and \( du = dw \) to get

\[ \int_{-9}^{11} \frac{1}{u^2 + 1} \, du \]

and use the fact that this is arctangent to see reduce to \( \tan^{-1}(11) - \tan^{-1}(9) \).
3. [20 points] A spherical tank of water is buried just below the ground. The sphere has radius $R$ feet (for some number $R > 2$), and has water 1 foot deep at the bottom of the tank. HINT: If you get stuck, recall that a sphere of radius $R$ can be formed by spinning the equation $y = \sqrt{R^2 - x^2}$ from $x = -R$ to $R$ around the $x$ axis.

a. [10 points] What is the volume of the water in the tank? (Your answer will be in terms of $R$.)

Solution: By slicing horizontally, one gets circles of water of depth $\Delta y$ as $y$ goes from the bottom of the tank to 1 foot higher. At height $y$ we can see from the picture and pythagorean theorem, that the circle has radius $\sqrt{R^2 - (R - y)^2}$. So the surface area of the slice is $\pi(2Ry - y^2)$. Then when we add up the volumes it becomes:

$$\int_{0}^{1} \pi(2Ry - y^2)dy = \pi(R - 1/3) \text{ft}^3.$$ 

(See image)

b. [10 points] How much work does it take to pump all the water out over the top of the tank? The water must only get to the top of the tank to be out. (Recall that water weighs 64.2 pounds per ft$^3$.)(Note that your answer will again be in terms of $R$.)

Solution: We slice the same way again. This time, we not only have that the slice of water has volume $\pi(2Ry - y^2)\Delta y$, but that it must be moved $R - y$ feet. Thus the work to lift this water out of the tank is $64.2\pi(2Ry - y^2)(R - y)\Delta y$ foot-pounds.

We again sum our slices getting:

$$\int_{0}^{1} 64.2\pi(2Ry - y^2)(R - y)dy = 64.2\pi \int_{0}^{1} 2R^2y - Ry^2 - 2Ry^2 + y^3 dy = 64.2\pi(R^2 - R + 1/4) \text{foot-pounds}$$
4. [27 points] Let $R$ be the region enclosed on top by $y = 1$ and on the bottom by the curves $y = 2/x$ and $y = x/4$ (see diagram).

a. [9 points] Set up an integral (or sum of integrals) equal to the size of the area $R$. DO NOT EVALUATE IT.

Solution: We can slice our $y$'s and get that from $y = \frac{1}{\sqrt{2}}$ to $y = 1$ we’re integrating $4y - 2/y$. So this

$$\int_{1/\sqrt{2}}^{1} 4y - 2/y \, dy.$$ 

b. [9 points] Set up an integral (or sum of integrals) equal to the volume of the object formed if $R$ were rotated around the $x$-axis. DO NOT EVALUATE IT

Solution: We need the volume of the largest part $\int_{1}^{4} \pi dx$ minus the volume of the smaller parts $\int_{2\sqrt{2}}^{1} \pi (x/4)^2 dx + \int_{2\sqrt{2}}^{4} \pi (2/x)^2 dx$.

c. [9 points] Set up an integral (or sum of integrals) equal to the volume of the object formed if $R$ were rotated around the $y$-axis. DO NOT EVALUATE IT

Solution: Again we take the larger volume: $\int_{1/\sqrt{2}}^{1} \pi (4y)^2 dy$ minus the smaller one $\int_{1/\sqrt{2}}^{1} \pi (2/y)^2 dy$. 
5. [12 points]
   a. [6 points] Describe and sketch a curve which has arc length equal to \( \int_{0}^{10} \sqrt{1 + x} \, dx \). Briefly show your work.
   
   Solution: The arc length formula is \( \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \) so we need \((f'(x))^2 = x\) so that \(f'(x) = \sqrt{x}\) and \(f(x) = \frac{2}{3} x^{3/2} + C\). Then it is just the length of this curve from \(x = 0\) to \(x = 10\).

   b. [6 points] Describe and sketch a region which has area equal to \( \int_{0}^{10} \sqrt{1 + x} \, dx \). Briefly show your work.
   
   Solution: The integral of a function is equal to the area under the curve, so it’s just the area from \(x = 0\) to \(x = 10\) under the curve \(\sqrt{1 + x}\).

6. [1 points] What are you doing to relax this weekend?
   
   Solution: Sleeping.