Math 106C: Fall 2012
Exam 1: September 28

Correct answers accompanied by incorrect or incomplete work will not receive full credit.

Formulas:

\[ |I - L_n| \leq \frac{K_1(b-a)^2}{2n} \]
\[ |I - R_n| \leq \frac{K_1(b-a)^2}{2n} \]
\[ |I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} \]
\[ |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2} \]

1. (10 points each) Evaluate the following integrals.

(a) \[ \int \frac{\sin(\ln x)}{x} \, dx \]
\[ u = \ln x \]
\[ du = \frac{1}{x} \, dx \]
\[ = \int \sin u \, du = -\cos u + C = \left[ -\cos (\ln x) + C \right] \]

(b) \[ \int \frac{5 \sin(2x)}{\cos^2(2x)} \, dx \]
\[ u = \cos (2x) \]
\[ du = -2 \sin (2x) \, dx \]
\[ -\frac{1}{2} \, du = \sin (2x) \, dx \]
\[ = \int -\frac{5}{2} \frac{1}{u^2} \, du = -\frac{5}{2} \int u^{-2} \, du = -\frac{5}{2} \cdot (-1) \cdot u^{-1} + C \]
\[ = \frac{5}{2} \cdot \frac{1}{\cos (2x)} + C = \left[ \frac{5}{2 \cos (2x)} + C \right] \]
2. (8 points each) Determine whether the following statements are TRUE or FALSE. Justify your answers.

(a) \[ \int_0^1 \frac{e^x}{1+e^x} \, dx = \int_1^2 \frac{du}{1+u} \]

\[ u = e^x \]

\[ du = e^x \, dx \]

\[ = \int_0^1 \frac{du}{1+u} \]

where \[ a = e^0 = 1 \]

\[ b = e^1 = e \]

So \[ \boxed{\text{FALSE}} \]

(b) If \( f(x) \) is increasing then the estimates of \( I = \int_a^b f(x) \, dx \) satisfy \( L_{2012} \leq M_{2012} \leq R_{2012} \).

\[ \boxed{\text{TRUE}} \]

If \( f(x) \) is increasing then for all intervals \( f(x_i) < f(m_i) < f(x_{i+1}) \). So

\[ f(x_i) \Delta x < f(m_i) \Delta x < f(x_{i+1}) \Delta x \]

and \[ \sum_{i=0}^{n-1} f(x_i) \Delta x < \sum_{i=0}^{n-1} f(m_i) \Delta x < \sum_{i=0}^{n-1} f(x_{i+1}) \Delta x \]

and \[ L_n < M_n < R_n \].

(c) If \( I = \int_1^8 f(x) \, dx \) and on the interval \([1, 8]\) we have both \(-5 \leq f'(x) \leq 4\) and \(-3 \leq f''(x) \leq 2\), then \( |I - T_7| \leq \frac{21}{12} \).

Use \[ |I - T_7| \leq \frac{k_2 (8-1)^3}{12 (\frac{8}{7})^2} \]

where \( k_2 \) is such that \( |f''(x)| \leq k_2 \)

We can take \( k_2 = 3 \).

So \[ |I - T_7| \leq \frac{3(7)}{12} = \frac{21}{12} \]

So \[ \boxed{\text{TRUE}} \]
3. Let \( t \) be hours after noon and let \( D(t) \) represent the rate at which the temperature of an object is changing measured in degrees per hour. \( D(t) \) is measured and the results put into a table as follows:

<table>
<thead>
<tr>
<th>( t )</th>
<th>3/2</th>
<th>5/3</th>
<th>11/6</th>
<th>2</th>
<th>13/6</th>
<th>7/3</th>
<th>5/2</th>
<th>8/3</th>
<th>17/6</th>
<th>3</th>
<th>19/6</th>
<th>10/3</th>
<th>7/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D(t) )</td>
<td>5.5</td>
<td>7.5</td>
<td>8.7</td>
<td>9.0</td>
<td>8.6</td>
<td>7.4</td>
<td>5.4</td>
<td>2.6</td>
<td>-1.0</td>
<td>-5.4</td>
<td>-10.6</td>
<td>-16.6</td>
<td>-23.4</td>
</tr>
</tbody>
</table>

(a) (4 points) What does the integral \( \int_{2}^{3} D(t) \, dt \) represent?

The net change in temperature from 2pm to 3pm.

(b) (12 points) Find each of the following estimates of \( \int_{2}^{3} D(t) \, dt \) using only the information in the table. If the table doesn't supply the information needed, explain why not.

- \( T_3 \)
  \[ \Delta x = \frac{3-2}{3} = \frac{1}{3} \]

  \[ T_3 = \left( \frac{1}{2} \right) \left( \frac{1}{3} \right) \left( 9.0 + 2 \left( \frac{7}{4} \right) + 2 \left( \frac{6}{4} \right) + (-5,4) \right) = \frac{309}{3} \]

- \( M_6 \)
  \[ \Delta x = \frac{3-2}{6} = \frac{1}{6} \]
  
  \[ \text{can't be done} \]

We would need the value of \( D(6) \) at the midpoints of intervals such as \( [2, \frac{13}{6}] \), information which we don't have.
4. Let $R$ be the region bounded by the curves $y = x^2 - 4$ and $y = 2x - 4$. (The region is shown below.)

(a) (10 points) Set up, but do not evaluate the integral(s) that you would use to find the area of $R$.

\[
\text{Area} = \int_0^2 \left(\text{top} - \text{bottom}\right) \, dx = \int_0^2 \left[ (2x - 4) - (x^2 - 4) \right] \, dx
\]

(b) (10 points) Set up, but do not evaluate the integral(s) that you would use to find the volume of the solid formed by rotating $R$ around the line $x = \frac{\pi}{2}$.

\[
\text{Vol} = \pi \int_{-4}^0 \left[ (r_{\text{out}})^2 - (r_{\text{in}})^2 \right] \, dy
\]

\[
= \pi \int_{-4}^0 \left[ \left(7 - \frac{y+4}{2}\right)^2 - \left(7 - \sqrt{y+4}\right)^2 \right] \, dy
\]
5. (10 points) Find a function whose arc length is given by \( \int_1^b \sqrt{1 + \left[ f'(x) \right]^2} \, dx \). Justify your answer.

arc length formula: \( \int_1^b \sqrt{1 + \left[ f'(x) \right]^2} \, dx \)

So \( \left[ f'(x) \right]^2 = \frac{1}{x^4} \) \( \Rightarrow \ f'(x) = \frac{1}{x^2} \) (or \( f'(x) = -\frac{1}{x^2} \))

\( \Rightarrow f(x) = \int f'(x) \, dx = \int \frac{1}{x^2} \, dx = -\frac{1}{x} + C \)

So one correct answer is \( \boxed{f(x) = -\frac{1}{x}} \)

6. (10 points) Consider a 15 ft tall conical tank with base radius of 5 ft and oriented as below. Suppose the tank is filled to 4 ft below the rim with benzene weighing 56 lb/ft\(^3\). Set up, but do not evaluate the integral representing the work done pumping the benzene to a level 3 ft above the top of the tank.

\[ \text{Vol} \text{ slab} = \pi \left( \frac{1}{3} y \right)^2 dy \]

\[ \text{Weight} \text{ slab} = \frac{\pi}{9} y^2 (56) dy \]

Work to move slab = \( (18-y) \frac{56\pi}{9} y^2 dy \)

Work to empty tank = \( \int_0^4 \frac{56\pi}{9} (18-y) y^2 dy \)