NAME:

Instruction: Read each question carefully. Explain ALL your work and give reasons to support your answers.
Advice: DON’T spend too much time on a single problem.

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1. Let \( P \) be the parallelogram in \( \mathbb{R}^3 \) with vertices
\[
A = (1, -1, 2), \quad B = (2, 0, 1), \quad C = (3, 2, -1), \quad \text{and} \quad D = (2, 1, 0).
\]
[Don’t spend too much time drawing the picture!]

(8 pts) (i) Find the area of \( P \).

The vector \( \vec{AB} \) is \((1, 1, -1)\) and \( \vec{CD} = (1, 1, -1) \) so \( AB \) is parallel to \( CD \). In other words, \( AB \) and \( AD \) are two adjacent edges of the parallelogram at the corner \( A \). Thus the area of \( P \) is given by
\[
\text{area}(P) = \| \vec{AB} \times \vec{AD} \|
\]
\[
= \begin{vmatrix}
i & j & k \\
1 & 1 & -1 \\
1 & 2 & -2 \\
\end{vmatrix}
\]
\[
= \| j + k \| = \sqrt{2}.
\]

(8 pts) (ii) Let \( E = (2, -2, 5) \). Find the volume of the parallelepiped spanned by the vectors \( \vec{AB}, \vec{AD} \) and \( \vec{AE} \).

The vector \( \vec{AE} \) is given by \((1, -1, 3)\). From (i), we have \( \vec{AB} \times \vec{AD} = j + k \). It follows that the volume of the desired parallelepiped is given by
\[
| \vec{AE} \cdot (\vec{AB} \times \vec{AD}) | = |(1)(0) + (-1)(1) + (3)(1)| = 2.
\]
2. Consider the following two planes \( P_1 : 4x - y + z = 2 \) and \( P_2 : 2x - z = 3 \) in \( \mathbb{R}^3 \).

(5 pts) (i) Find a point on the line of intersection between the planes \( P_1 \) and \( P_2 \).

There are infinitely many points that lie on the line of intersection. In particular, when \( z = 0 \), the line \( 4x - y = 2 \) lies on \( P_1 \); the line \( 2x = 3 \) lies on \( P_2 \) and thus \( x = \frac{3}{2} \). On the line \( 4x - y = 2 \), \( y = 4 \). Hence, the point \( \left( \frac{3}{2}, 4, 0 \right) \) lies on both \( P_1 \) and \( P_2 \).

(3 pts) (ii) Find a vector orthogonal to the plane \( P_1 \).

Since \( 4x - y + z = 2 \), we have \( 4(x - 0) + (-1)(y - 0) + 1(z - 2) = 0 \) so that \( (4, -1, 1) \) is orthogonal to \( P_1 \).

(3 pts) (iii) Find a vector orthogonal to the plane \( P_2 \).

Similar to (ii), \( (2, 0, -1) \) is orthogonal to \( P_2 \).

(6 pts) (iv) Find a parametrization for the line of intersection between \( P_1 \) and \( P_2 \).

From (i), \( x_0 = (\frac{3}{2}, 4, 0) \) is on the line of intersection. The vector
\[
(4, -1, 1) \times (2, 0, -1) = (1, 6, 2)
\]
is in the same direction as the line of intersection. Thus,
\[
x = (\frac{3}{2}, 4, 0) + t(1, 6, 2)
\]
is a parametrization of the desired line.
3. The position of a particle in $\mathbb{R}^3$ is given by the parametrization 

$$r(t) = e^t \mathbf{i} + \mathbf{j} + \sin tk,$$ 

for $t \geq 0$.

(5 pts) (i) Find the velocity $v(t)$ of the particle at time $t$.

$$v(t) = r'(t) = e^t \mathbf{i} + 0\mathbf{j} + \cos tk.$$ 

(6 pts) (ii) Find the projection of the initial velocity in the direction of $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$.

**Initial velocity** $v(0) = \mathbf{i} + \mathbf{k}$. Then

$$\text{proj}_a v(0) = \frac{v(0) \cdot \mathbf{a}}{||\mathbf{a}||^2} \mathbf{a} = \frac{1}{5} (1 + 2\mathbf{j}).$$

(6 pts) (iii) Find the angle between the position and the velocity of the particle at $t = \frac{\pi}{4}$.

First, $r \left( \frac{\pi}{4} \right) = (e^{\frac{\pi}{4}}, 1, \frac{\sqrt{2}}{2})$ and $v \left( \frac{\pi}{4} \right) = (e^{\frac{\pi}{4}}, 0, \frac{\sqrt{2}}{2})$. It follows that

$$r \left( \frac{\pi}{4} \right) \cdot v \left( \frac{\pi}{4} \right) = e^{\frac{\pi}{4}} + \frac{1}{2}.$$ 

On the other hand, the dot product above is equal to $||r|| ||v|| \cos \theta$ where $\theta$ is the angle between the position and the velocity. A straightforward computation shows that

$$\theta = \arccos \left( \frac{e^{\frac{\pi}{4}} + \frac{1}{2}}{e^{\frac{\pi}{4}} + \frac{3}{2}} \right).$$
4. (4 pts) (i) Consider the line given by the parametric equations \( x = t + 1, y = 2 - 3t, z = 2t + 1 \). Write this line in vector form \( \mathbf{x} = t\mathbf{m} + \mathbf{x}_0 \)

\[ \mathbf{x} = t(1, -3, 2) + (1, 2, 1). \]

(8 pts) (ii) Give an equation for the plane that is perpendicular to the line given in (i) and that contains the point \((1, 1, 0)\). [Hint: use (i)]

Since the point \((1, 1, 0)\) is on the plane, the vector \( \mathbf{x} - (1, 1, 0) \) is parallel to the plane where \( \mathbf{x} = (x, y, z) \) denotes an arbitrary point on the plane. Moreover, the vector \((1, -3, 2)\) is orthogonal to this plane. Thus,

\[ (\mathbf{x} - (1, 1, 0)) \cdot (1, 3, 2) = 0 \]

or equivalently,

\[ x - 3y + 2z + 2 = 0. \]

(5 pts) (iii) Sketch the vector field \( f(x, y) = (xy, 2) \) at the points \( A = (1, -1), B = (0, 1), C = (-1, -1), D = (1, 1) \).
5. Let \( f(x, y) = xy. \)

(5 pts) (i) Sketch the level curves of \( f(x, y) = 3 \) and \( f(x, y) = -1. \)

\[
\begin{align*}
\text{(i,3)} & \quad f(x,y)=3 \\
(1,-1) & \quad f(x,y)=-1
\end{align*}
\]

(5 pts) Describe or sketch the level set \( h(x, y, z) = 4 \) where \( h(x, y, z) = x^2 + \frac{y^2}{9}. \)

Since \( h(x, y, z) = x^2 + \frac{y^2}{9}, \) the level set \( h(x, y, z) = 4 \) is simply \( x^2 + \frac{y^2}{9} = 4. \) Since this equation holds for ANY value of \( z, \) it follows that the level set (surface) is an infinite elliptic cylinder.

(6 pts) (iii) What are the cylindrical coordinates \((r, \theta, w)\) AND the spherical coordinates \((\rho, \theta, \phi)\) of the point \((1,1,\sqrt{2})?\)

For cylindrical coordinates, we have \( x = r \cos \theta, y = r \sin \theta, w = z. \) Note that \( r^2 = x^2 + y^2 = 1 + 1 = 2 \) so \( r = \sqrt{2}. \) Moreover, \( 1 = \sqrt{2} \cos \theta \) which implies that \( \theta = \frac{\pi}{4}. \) The cylindrical coordinates of the point \((1,1,\sqrt{2})\) are \((\sqrt{2}, \frac{\pi}{4}, \sqrt{2}).\) For the spherical coordinates, \( \rho^2 = x^2 + y^2 + z^2 = 4 \) or \( \rho = 2. \) Since \( \theta = \frac{\pi}{4} \) and \( z = \rho \cos \phi = \sqrt{2}, \) we have \( \phi = \frac{\pi}{4}. \) The spherical coordinates of the point \((1,1,\sqrt{2})\) are \((2, \frac{\pi}{4}, \frac{\pi}{4}).\)
6. Let $T$ be a linear transformation given by $T(x_1, x_2) = (3x_2 - x_1, 2x_1)$.

(7 pts)(i) Find the matrix $A$ associated to $T$ such that $T(x) = Ax$.

The matrix $A$ associated with the linear transformation $T$ is given by

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}.$$

(10 pts)(ii) Suppose $W$ is a figure in $\mathbb{R}^2$ so that $T(W)$ is the figure shown below.

What is the area of the figure $W$? [Hint: $T(W)$ is made up of a parallelogram and a rectangle; what is the relation between the area of $W$ and the area of $T(W)$?]

Since $T(W)$ is a union of two parallelograms, then so is $W$. It follows that

$$\text{area}(T(W)) = |\det A| \cdot \text{area}(W).$$

Now, $T(W)$ is the sum of the two areas which is $(3)(4) + (4)(5) = 32$ and $\det A = -6$. Thus, $\text{area}(W) = \frac{32}{6}$. 