Math 106: Review for Exam I - SOLUTIONS

1. Find the following. [Substitution tip: usually let $u$ be a function that’s “inside” another function, especially if $du$ (possibly off by a multiplying constant) is also present in the integrand.]

(a) Let $u = \sqrt{x}$, so $du = \frac{dx}{2\sqrt{x}}$ and $2\, du = \frac{dx}{\sqrt{x}}$

\[
\int_1^4 e^{\sqrt{x}} \frac{dx}{\sqrt{x}} = \int_{x=1}^{x=4} e^u \cdot 2 \, du \quad \text{If you prefer to switch the limits, use } u = 1 \text{ to } u = 2.
\]

\[
= 2e^u \bigg|_{x=1}^{x=4} = 2e^{\sqrt{4}} - 2e^1 = 2e^2 - 2e \approx 9.342
\]

(b) Let $u = \cos(5x)$, so $du = -5 \sin(5x)$ and $-\frac{du}{5} = \sin(5x)$.

This time, we’ll change the limits:

$x = \pi \Rightarrow u = \cos(5 \cdot \pi) = -1$ and $x = 2\pi \Rightarrow u = \cos(5 \cdot 2\pi) = 1$

\[
\int_{\pi}^{2\pi} \cos(5x) \sin(5x) \, dx = \int_{-1}^{1} u^7 \cdot -\frac{du}{5}
\]

\[
= -\frac{1}{5} \int_{-1}^{1} u^7 \, du = -\frac{1}{5} \left. \frac{u^8}{8} \right|_{-1}^{1} = -\frac{1}{40} \left[ \frac{1}{8} - \frac{(-1)^8}{8} \right] = 0
\]

(c) Use $u = x^3$, so $du = 3x^2 \, dx$ and $\frac{du}{3} = x^2 \, dx$.

\[
\int \frac{7x^2}{1 + x^6} \, dx = 7 \int \frac{du}{3} \frac{du}{1 + u^2} = \frac{7}{3} \arctan u + C = \frac{7}{3} \arctan(x^3) + C
\]
(d) Use \( u = 10 - x \), so \( du = -dx \) and \( dx = -du \).
We'll change the limits:
\[
x = 6 \Rightarrow u = 10 - x = 4 \quad \text{and} \quad x = 10 \Rightarrow u = 10 - 10 = 0
\]
\[
\int_6^{10} x\sqrt{10 - x} \, dx = \int_4^0 (10 - u)\sqrt{u}(-du) \quad \text{Since} \ u = 10 - x, \ \text{we know} \ x = 10 - u.
\]
\[
= \int_4^0 (u - 10)\sqrt{u} \, du
= \int_4^0 (u^{3/2} - 10u^{1/2}) \, du
= \left[ \frac{2}{5} u^{5/2} - \frac{20}{3} u^{3/2} \right]_4^0
= (0 - 0) - \left( \frac{2}{5} 4^{5/2} - \frac{20}{3} 4^{3/2} \right)
= \frac{608}{15} = 40.53
\]

2. If \( f(x) \) is decreasing and concave up, put the following quantities in ascending order.
\[
L_{100}, R_{100}, T_{100}, M_{100}, \int_a^b f(x) \, dx
\]
\[
R_{100} < M_{100} < \int_a^b f(x) \, dx < T_{100} < L_{100}
\]
What can you say with certainty about where \( S_{200} \) would fit into your list above?
It would be somewhere between \( M_{100} \) and \( T_{100} \) but we don’t know how it compares to \( \int_a^b f(x) \, dx \).

3. Suppose \( f(t) \) is the rate of change (in animals per month) of a population \( P(t) \).
(a) What does \( \int_4^{12} f(t) \, dt \) represent in this problem?
It represents the total (or net) change in the number of animals during the time period \([4, 12] \).
(b) Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_4^{12} f(t) \, dt \) given the data in the table below.

<table>
<thead>
<tr>
<th>( t )</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

\[
L_4 = (15 + 11 + 8 + 4)(2) = 76 \quad \text{and} \quad R_4 = (11 + 8 + 4 + 3)(2) = 52
\]
\[
T_4 = \frac{L_4 + R_4}{2} = 64
\]
\[
M_2 = (11 + 4)(4) = 60 \quad \text{and} \quad T_2 = \frac{L_2 + R_2}{2} = \frac{(15 + 8)(4) + (8 + 3)(4)}{2} = 68.
\]
\[
S_4 = \frac{2M_2 + T_2}{3} = \frac{2(60) + 68}{3} = \frac{188}{3} = 62.67
\]

4. Find bounds for each of the following errors if \( I = \int_2^7 \ln x \, dx \).
(a) \( |I - L_{100}| \leq \frac{K_1(b - a)^2}{2n} = \frac{1}{2}(7 - 2)^2}{2(100)} = \frac{1}{16}
\]
\[
K_1 = \max |f'(x)| \text{ on } [2, 7] = \max \frac{1}{x} \text{ on } [2, 7] = \frac{1}{2} \text{ (occurs at } x = 2)\]
(b) $|I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12(100)^2} = \frac{1}{3840}$

$K_2 = \text{max of } |f''(x)| \text{ on } [2,7] = \text{max of } \frac{1}{x^2} \text{ on } [2,7] = \frac{1}{4} \text{ (occurs at } x = 2)\]

(c) $|I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{\frac{1}{4}(7-2)^3}{24(100)^2} = \frac{1}{7680}$

$K_2 = \text{same as in previous part}$

5. If $I = \int_2^7 \ln x \, dx$, how many subdivisions are required to obtain a trapezoidal sum approximation with error of at most $1/1,000,000$?

From part (b) above, we know that $|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{\frac{1}{4}(7-2)^3}{12n^2} = \frac{125}{48n^2}$.

Thus, we want $\frac{125}{48n^2} \leq \frac{1}{1,000,000}$.

Multiplying each side by $1,000,000n^2$ gives $\frac{125,000,000}{48} \leq n^2$.

Taking the square root of each side results in $\sqrt{\frac{125,000,000}{48}} \leq n$.

Since $\sqrt{\frac{125,000,000}{48}} = 1613.743...$, we must at least 1614 subdivisions.

6. Write integrals equal to

(a) the arc length of $y = x^2$ on the interval $[1,5]$

arc length of $y = f(x)$ on $[a,b] = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx = \int_1^5 \sqrt{1 + (2x)^2} \, dx \approx 24.395$

(b) the area bounded by $y = x^2 - 8x + 24$ and $y = 3x$

First, find where the curves intersect.

\[ x^2 - 8x + 24 = 3x \]
\[ x^2 - 11x + 24 = 0 \]
\[ (x - 3)(x - 8) = 0 \]
\[ \Rightarrow x = 3, x = 8 \]

Between $x = 3$ and $x = 8$, $y = 3x$ is above $y = x^2 - 8x + 24$. (Plug in $x = 5$ or graph to check.)

So, the area between them is

\[ \int_3^8 [3x - (x^2 - 8x + 24)] \, dx. \]

[This equals 125/6.]

7. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$. Write an integral equal to the volume generated if this region is revolved about

(a) the $x$-axis
volume of slice \( \approx \pi r^2 \Delta x \)
\( = \pi y^2 \Delta x \)
\( = \pi (\sqrt{x})^2 \Delta x \)
\( = \pi x \Delta x \)

total volume = \( \pi \int_0^9 x \, dx \)

(b) the line \( x = -1 \)

volume of slice \( \approx \pi R^2 \Delta y - \pi r^2 \Delta y \)
\( = \pi (10^2) \Delta y - \pi (1 + x)^2 \Delta y \)
\( = \pi [100 - (1 + y^2)^2] \Delta y \)

total volume = \( \pi \int_0^3 [100 - (1 + y^2)^2] \, dy \)

8. A pyramid has a square base 30 feet to a side and a height of 10 feet. Write integrals equal to

(a) the volume of the pyramid

We slice horizontally, so each slice is a “box” with a square top and bottom and a height (thickness) of \( \Delta h \),

The picture shown below is a vertical cross-section through the center of the pyramid.

volume of slice \( \approx s^2 \Delta h \approx [3(10-h)]^2 \Delta h \) From above.
weight of slice \( \approx 62.4 [3(10-h)]^2 \Delta h \) Weight=(density)(volume).
work to lift slice \( \approx 62.4 [3(10-h)]^2 \Delta h (15-h) \) Work=(force)(distance); here, force=weight.

total work = \( 62.4 \int_0^8 [3(10-h)]^2 (15-h) \, dh \)