Math 106 - Quiz 2 - September 18, 2006

Instructions: Show all of your work and circle your final answers. Calculators are allowed, but notes and books are not.

1. (10 pts.) Consider the integral \( I = \int_{0}^{2} e^{-x^2} \, dx \). Using a Riemann sum with right endpoints, how many rectangles would we need to guarantee accuracy to within 0.001 of the actual value of \( I \)?

\[
| I - R_n | \leq \frac{K_1 (b-a)^2}{2n}, \quad \text{where} \quad | f'(x) | \leq K_1, \quad \text{for all} \quad x \in [a, b].
\]

\( f(x) = e^{-x^2} \). So \( f'(x) = -2x e^{-x^2} \).

Graphically, we see that | f'(x) | < 1.

So we choose \( K_1 = 1 \). (A choice of \( K_1 = 0.6 \) also works, and it gives a better answer.)

So \( | I - R_n | \leq \frac{1 (2-0)^2}{2n} \), and we need this to be less than 0.001.

\[
\frac{1 (2-0)^2}{2n} < 0.001.
\]

So \( \frac{2}{n} < 0.001 \). So \( 2000 < n \).

Thus, with \( n = 2001 \) intervals, we have the desired accuracy.

2. (10 pts.) Consider the initial value problem \( y' = y - 2, y(0) = 1 \).

(a) By hand, use Euler's method with 2 steps of size 1 to estimate \( y(2) \).

(b) Is \( y(t) = 2 - e^t \) a solution to this initial value problem? Explain.

\( a) \Delta t = 1 \quad t_0 = 0, \quad y_0 = 1 \quad \rightarrow \quad y' = y - 2 = 1 - 2 = -1 \quad \text{slope at } (0, 1). \)

\( \Delta y = \text{slope} \cdot \Delta t = -1 \cdot 1 = -1. \)

So \( y_1 = y_0 + \Delta y = 1 + (-1) = 0 \)

\( t_1 = t_0 + \Delta t = 0 + 1 = 1. \quad \text{So } y(1) \approx 0 \)

\( t_1 = 1, \quad y_1 = 0 \quad \rightarrow \quad y' = y - 2 = 0 - 2 = -2 \)

\( \Delta y = \text{slope} \cdot \Delta t = -2 \cdot 1 = -2. \)

So \( y_2 = y_1 + \Delta y = 0 + (-2) = -2 \)

\( t_2 = t_1 + \Delta t = 1 + 1 = 2 \).

\( \text{So } y(2) \approx -2. \)

\( b) \) If \( y(t) = 2 - e^t \), then \( y'(t) = \frac{d}{dt} (2 - e^t) = -e^t. \) And \( y - 2 = (2 - e^t) - 2 = -e^t \).

Thus, \( y' = y - 2 \). Also, \( y(0) = 2 - e^0 = 0 = -1 = 2 - 1 = 1 \). So \( y = 2 - e^t \) is a solution to the IVP.