This page contains useful information for problem 1.

The Reduced Row Echelon Form (rref) of

\[
\begin{bmatrix}
2 & -4 & -4 & -2 & 6 \\
-1 & 2 & 2 & 1 & -3 \\
4 & -3 & 7 & 1 & 1 \\
1 & 1 & 7 & 2 & 4
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 6 & 4 & 5 \\
-1 & 0 & -3 & -2 & 3 \\
1 & -1 & 5 & 7 & 5 \\
3 & 7 & -5 & 9 & 3
\end{bmatrix}
\]

is

\[
\begin{bmatrix}
1 & 0 & 4 & 1 & 0 \\
0 & 1 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1/2 & 1 & -1/2 & 3 & 0 \\
1/2 & 2 & -5/2 & 2 & 0 \\
1/2 & 1 & -1/2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The Reduced Row Echelon Form (rref) of

\[
\begin{bmatrix}
2 & 0 & 6 & 4 \\
-1 & 0 & -3 & -2 \\
1 & -1 & 5 & 7 \\
3 & 7 & -5 & 9
\end{bmatrix}
\begin{bmatrix}
2 & -4 & -4 & -2 & 6 \\
-1 & 2 & 2 & 1 & -3 \\
4 & -3 & 7 & 1 & 1 \\
1 & 1 & 7 & 2 & 4
\end{bmatrix}
\]

is

\[
\begin{bmatrix}
1 & 0 & 3 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & -2 & -6 & -2 & 4 & 0 \\
-1/2 & 1 & 1/2 & -1/2 & 0 & 3/2 \\
1/2 & 0 & 2 & -1/2 & 0 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The RREF of the transpose of \( P \) in problem 1 is

\[
\begin{bmatrix}
1 & -1/2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
1. Use the sheet of Row-Reduced Matrices (page “A”) to help answer these questions. You shouldn’t need a calculator in any part of problem 1.

Let \( P = \begin{bmatrix} 2 & -4 & -4 & -2 & 6 \\ -1 & 2 & 2 & 1 & -3 \\ 4 & -3 & 7 & 1 & 1 \\ 1 & 1 & 7 & 2 & 4 \end{bmatrix} \) and \( Q = \begin{bmatrix} 2 & 0 & 6 & 4 \\ -1 & 0 & -3 & -2 \\ 1 & -1 & 5 & 7 \\ 3 & 7 & -5 & 9 \end{bmatrix} \).

Label the column vectors of \( P \) as \( p_1, \ldots, p_5 \), and those of \( Q \) as \( q_1, \ldots, q_4 \).

1A. Find a basis \( B \) for \( \text{Col}(P) \). Write your answer using the symbols \( p_1, \ldots, p_5 \) (don’t write out the actual column vectors).

1B. Find a basis \( C \) for \( \text{Col}(Q) \), using the symbols \( q_1, \ldots, q_4 \).

1C. Let \( v = \begin{bmatrix} 6 \\ -3 \\ 4 \\ 21 \end{bmatrix} \). Use the RREF sheet to completely solve \( P \mathbf{x} = \mathbf{v} \); write your answer in the “\( \mathbf{x} = \mathbf{p} + \mathbf{v}_h \)” notation.

1D. Find both \( \mathbf{v}_B \) and \( \mathbf{v}_C \); clearly identify which is which. This is not a change of basis problem.
(Problem 1 continues here)

1E. In terms of the symbols \( p_1, \ldots, p_5 \), and \( q_1, \ldots, q_4 \), what superaugmented matrix represents the problem of expressing each of the basis vectors in \( B \) as linear combinations (LC's) of those in \( C \)?

1F. Explicitly find the \text{rref} of the matrix in 1E. (The info you need is on the RREF sheet!)

1G. Explicitly give the change of basis matrix from \( B \) to \( C \).

1H. Use the \text{rref} of \( P \) to find a basis (call it \( D \)) for \( \text{Row}(P) \).

1I. Use \( P^T \) and its \text{rref} to find another basis (call this one \( E \)) for \( \text{Row}(P) \).
Problem 1 continues here.

1J. Let \( r_4 \) be the fourth row vector in \( P \). What is \( [r_4]_D \)? What is \( [r_4]_E \)? (Identify which answer is which).

1K. Find a basis for \( \text{Nul}(P) \).

1L. Find a basis for \( \text{Col}(P)^\perp \).
2. Let \( A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 2 \\ -1 & -1 & 4 \end{bmatrix} \)

2A. Find in factored form, the characteristic polynomial of \( A \).

2B. State the eigenvalues of \( A \) and their multiplicities.
3. Let \( C = \begin{bmatrix} 5 & 0 & -1 \\ 3 & 4 & -3 \\ 5 & 0 & -1 \end{bmatrix} \).

Facts: (1) \( C \) is not invertible (2) The vector \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) is an eigenvector of \( C \). (3) \( C \) is diagonalizable.

Find \( P \) and \( D \) such that \( P \) is invertible, \( D \) is diagonal, the columns of \( P \) are eigenvectors for \( C \), and \( C = PDP^{-1} \). Show all your work, including any matrices you require, both before and after row-reduction.
4. There is no parabola of the form \( \beta_2 x^2 + \beta_1 x + \beta_0 \) that contains all the points (3, 3), (2, 7), (0, 1) and (-1, 11). You do not need to verify this.

4A. Find the best-fit parabola of this form for these four points. Identify your design matrix, parameter vector, and observation vector. Show all your work.

4B. What are the four \( y \)-coordinates on the best-fit parabola corresponding to the \( x \) coordinates of the four points (3, 3), (2, 7), (0, 1) and (-1, 11)?

4C. What is the distance from the vector \( y = \begin{bmatrix} 3 \\ 7 \\ 1 \\ 11 \end{bmatrix} \) to the column space of the matrix \( \begin{bmatrix} 9 & 3 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \)?

4D. How does the answer to 4C relate to the points (3, 3), (2, 7), (0, 1) and (-1, 11) and the best-fit parabola from 4A?
5. Let $F$ be the vector space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Let $H$ be the subset of all members of $F$ which have a non-negative $y$-intercept. (The $y$-intercept of a function $f$ is the $y$-coordinate where the graph of $f$ crosses the $y$-axis).

For each part of the definition of subspace, show the $H$ satisfies that part (give a proof) or give an explicit counterexample that $H$ does not satisfy that part.
6. Define what it means for a set of vectors $S = \{v_1, \ldots, v_p\}$ to be *linearly independent*.

7. Define what it means for a transformation $T$ from $\mathbb{R}^k$ to $\mathbb{R}^j$ to be a *linear* transformation.

8. Suppose $P$ in problem 1 is the matrix of a Linear Transformation $T : \mathbb{R}^k \to \mathbb{R}^j$.
   8A. What are $k$ and $j$?

   $k = \quad j =$

   8B. Is $T$ a one-to-one linear transformation? Explain.

   8C. Is $T$ onto $\mathbb{R}^j$? Explain.