Math 105: Review for Final Exam, Part II - SOLUTIONS

1. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

So, we write an equation that relates $a$ and $b$ and then differentiate implicitly with respect to time $t$.

\[
2a \frac{da}{dt} + 0 = 2b \frac{db}{dt}
\]

\[
\frac{da}{dt} = \frac{b}{a} \frac{db}{dt}
\]

At the moment in question, $b = 10$, $a = 8$ (by the Pythagorean Theorem), and $\frac{db}{dt} = -3$.

So, $\frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75$ feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

2. You are watching a plane flying toward your position at a constant height of 3 miles and a speed of 500 miles per hour relative to the ground. At the moment when the plane is 5 miles from you (diagonally), at what rate is the angle of your vision toward the plane changing?

So, we write an equation that relates $a$ and $\theta$ and then differentiate implicitly with respect to time $t$.

\[
\tan \theta = \frac{3}{a}
\]

\[
\sec^2 \theta \frac{d\theta}{dt} = -\frac{3}{a^2} \frac{da}{dt}
\]

\[
\frac{d\theta}{dt} = -\frac{3}{a^2} \frac{da}{dt} \cos^2 \theta
\]

At the moment in question, $c = 5$, so by the Pythagorean Theorem we know that $a = 4$ and that $\cos \theta = \frac{4}{5}$.

So, $\frac{d\theta}{dt} = -\frac{3}{4^2} (-500) \left(\frac{4}{5}\right)^2 = 60$ radians per hour.

3. Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.

IVT: If $f$ is continuous on $[a, b]$ and $y$ is a number between $f(a)$ and $f(b)$, then there is a number $c$ between $a$ and $b$ such that $f(c) = y$.

For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between -2 and 3, the IVT says there is a number $c$ between 1 and 2 such that $f(c) = 0$; this $c$ is the desired root.
4. What (if anything) does the Extreme Value Theorem say about \( f(x) = x^2 \) on each of the following intervals?

EVT: If \( f \) is continuous on \([a, b]\), then \( f \) has both a maximum and a minimum on \([a, b]\).

(a) \([1, 4]\)

\( f \) has a maximum and a minimum on \([1, 4]\)

(b) \((1, 4)\)

The EVT doesn’t apply because \((1, 4)\) is not a closed interval since its endpoints are not included.

5. Find the value of the constant \( c \) that the Mean Value Theorem specifies for \( f(x) = x^3 + x \) on \([0, 3]\).

MVT: If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a number \( c \) between \( a \) and \( b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

For our function, we have \( f'(x) = 3x^2 + 1 \), so \( f'(c) = 3c^2 + 1 \).

So, we solve \( 3c^2 + 1 = 10 \), which means \( c = \sqrt{3} \). (The other solution, \( x = -\sqrt{3} \), is not in our interval \([0, 3]\).)

6. Water is leaking out of a tank at a decreasing rate \( r(t) \) as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

overestimate = \( L_4 = (15 + 11 + 8 + 4)(2) = 76 \)
underestimate = \( R_4 = (11 + 8 + 4 + 3)(2) = 52 \)

(b) Interpret the expression \( \int_2^6 r(t) \, dt \) in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval \([2, 6]\) minutes.

7. Consider the graph of \( f(t) \) shown. It is made of straight lines and a semicircle.

Let \( G(x) = \int_0^x f(t) \, dt \) and \( H(x) = \int_{-3}^x f(t) \, dt \).

(a) Compute \( G(2), G(4), G(-4), \) and \( H(4) \).

First, \( G(2) = \int_0^2 f(t) \, dt \) is the area under \( f \) between \( t = 0 \) and \( t = 2 \). This is a rectangle plus a triangle and has area \( 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3 \).
Similarly, $G(4) = \int_{0}^{4} f(t) \, dt = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi(1)^2 = 3 + \frac{\pi}{2}$.

Now, remembering that area below the $t$-axis counts as negative and that $\int_{b}^{a} f(t) \, dt = -\int_{a}^{b} f(t) \, dt$, we have 

$$G(-4) = -\int_{0}^{-4} f(t) \, dt = -\left[ -2 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi(1)^2 \right] = 4.$$  

Finally, $H(4) = \int_{-3}^{4} f(t) \, dt = -2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi(1)^2 = 1 + \frac{\pi}{2}$.

(b) Where is $G$ increasing? Where is $G$ decreasing?

For parts (b), (c), and (d), recall that we learned in class that $G' = f$.

$G$ is increasing where $f$ is positive: $(-1, 4]$. Note that $G$ has a horizontal slope at $x = 2$ but since $f$ is positive on each side of $t = 2$, we say $G$ is increasing at $x = 2$.

$G$ is decreasing where $f$ is negative: $[−4, −1)$.

(c) Where is $G$ concave up? Where is $G$ concave down?

$G$ is concave up where $f$ is increasing: $(-2, 0) \cup (2, 3)$.

$G$ is concave down where $f$ is decreasing: $(1, 2) \cup (3, 4]$.

(d) At what $x$-value(s) does $G$ have a local maximum? At what $x$-value(s) does $G$ have a local minimum?

$G$ has a local maximum where $f$ changes from positive to negative: never.

$G$ has a local minimum where $f$ changes from negative to positive: $x = -1$.

(e) Find a formula that relates $G$ and $H$.

From their definitions, $H(x) = \int_{-3}^{0} f(t) \, dt + G(x) = -2 + G(x)$.

(f) How would your answers to (b), (c), and (d) change if the questions were about $H$ instead of $G$?

They would not change at all because $H'(x) = G'(x)$.

8. (a) Use sigma notation to express $L_{10}$ and $M_{10}$ as approximations to $\int_{20}^{60} \ln x \, dx$.

We’re subdividing the interval into 10 pieces, so each piece has width $\Delta x = \frac{60 - 20}{10} = 4$.

$$L_{10} = [f(20) + f(24) + f(28) + ... + f(52) + f(56)] \Delta x = [\ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56)] \cdot 4 = \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4$$

$$M_{10} = [f(22) + f(26) + f(30) + ... + f(54) + f(58)] \Delta x = [\ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58)] \cdot 4 = \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4$$

(b) Draw a sketch that represents the sum $M_{4}$.

Now we’re subdividing the interval into 4 pieces, so each piece has width $\Delta x = \frac{60 - 20}{4} = 10$.

Note that the height of each rectangle is determined by the $y$-value of the curve at the middle $x$-value of the rectangle (that is, at $x = 25, 35, 45, 55$).
9. Find the following.

(a) all antiderivatives of $1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5} + \sec^2(6x) + \frac{7}{1+100x^2}$

Any such antiderivative will take the form $x + x^2 + \frac{x^4}{4} + 4\frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + \frac{\tan(6x)}{6} + \frac{7\arctan(10x)}{10} + C$.

Note that we have used the facts that $\sqrt{x} = x^{1/2}$ and $1/x^5 = x^{-5}$.

(b) $\int_{-2}^{2} \sqrt{4-x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi$  

This integral represents the area of a semicircle of radius 2.

(c) $\frac{d}{dx} \int_{1}^{x} \sin t \, dt = \sin x$  

The derivative of the area function is the original function.

(d) $\int_{0}^{2} x^2 \, dx = \frac{x^3}{3} \bigg|_{0}^{2} = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$. 