Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers.

1. Recall that the Taylor series for \( e^x \) centered at \( x_0 = 0 \) is \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \) on \( (-\infty, \infty) \).

   \( a \) (5 pts.) Find the first four non-zero terms of the Taylor series for \( f(x) = xe^{-2x} \) centered at \( x_0 = 0 \).

   \[
   e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
   e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots = 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k x^k}{k!} \\
   xe^{-2x} = x \left[ 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} + \cdots \right] = x - 2x^2 + 2x^3 - \frac{4x^4}{3!} + \frac{8x^5}{5!} + \cdots + \frac{(-1)^k 2^k x^{k+1}}{k!} + \cdots \\
   f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{k!} x^{k+1}
   \]

   \( b \) (5 pts.) What is \( f^{(13)}(0) \)? Justify your answer.

   The general formula for a Taylor series at \( x_0 = 0 \) is given by:

   \[
   f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots + \frac{f^{(13)}(0)}{13!} x^{13} + \cdots
   \]

   Notice that \( f^{(13)}(0) \) is part of the coefficient of \( x^{13} \), namely \( \frac{f^{(13)}(0)}{13!} \).

   From \( a \), the coefficient of \( x^{13} \) is \( \frac{(-1)^{12} 2^{12}}{12!} \).

   Equate these coefficients and solve for \( f^{(13)}(0) \):

   \[
   \frac{(-1)^{12} 2^{12}}{12!} \Rightarrow f^{(13)}(0) = \frac{13 \cdot 2^{13}}{12!} = 53248
   \]

2. (10 pts.) Consider the initial value problem \( \frac{dy}{dx} = x + y \) and \( y(2) = 1 \). Apply Euler’s method using step size \( \Delta x = 0.25 \) to estimate \( y(3) \). Round to 4 digits after the decimal. (Note: you are not required to use the table, regardless of your method, you need to show enough work to justify your answer.)

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
x & 2 & 2.25 & 2.50 & 2.75 & 3 \\
\hline
y & 1 & 1.75 & 2.75 & 4.0425 & 5.7656 \\
\hline
\frac{dy}{dx} & 3 & 2.25 + 1.75 & 2.50 & 6.8125 & \frac{dy}{dx} \\
\hline
\Delta y & 0.75 & 0.25 & 1 & 1.3125 & 1.7031 \\
\hline
\end{array}
\]

\( \Delta x = 0.25 \) Use \( \frac{dx}{dy} = x + y \) to find slope.

\( X_{k+1} = X_k + \Delta x \)

\( Y_{k+1} = Y_k + \Delta y \)

\( X_{new} \) \ 
\( Y_{new} \)

\( X_{old} \) \ 
\( Y_{old} \)

\( y_{k+1} = y_k + \Delta y \)

\( new x \) \ 
\( new y \)

\( dy \)
3. (8 pts each) Evaluate the following integrals.

(a) \[ \int_{0}^{1} \arcsin x \, dx \]

Let \( u = \arcsin x \), \( du = \frac{1}{\sqrt{1-x^2}} \, dx \)

\( v = x \)

Use integration by parts

\[ \int_{0}^{1} \arcsin x \, dx = x \arcsin x \bigg|_{0}^{1} - \int_{0}^{1} \frac{x}{\sqrt{1-x^2}} \, dx \]

Change limits: \( x = 0 \rightarrow \omega = 1 \); \( x = 1 \rightarrow \omega = 0 \)

This integral is improper since \( \omega^{\frac{1}{2}} \) is undefined at \( \omega = 0 \)

\[ = x \arcsin x \bigg|_{0}^{1} + \lim_{b \to 0} \int_{0}^{b} \omega^{\frac{1}{2}} \, d\omega \]

\[ = x \arcsin x \bigg|_{0}^{1} + \left[ \frac{2}{3} \omega^{\frac{3}{2}} \right]_{0}^{b} \]

\[ = x \arcsin x \bigg|_{0}^{1} + \left[ \frac{2}{3} \right] \left( \sqrt{b} - 0 \right) \]

\[ = (\arcsin 1 - 0) + \left( 0 - \frac{2}{3} \right) \]

\[ = \frac{\pi}{2} - 1 \]

(b) \[ \int \frac{\sqrt{x^2 - 9}}{x} \, dx \]

Let \( x = 3 \sec \theta \), \( dx = 3 \sec \theta \tan \theta \, d\theta \)

\[ \int \frac{\sqrt{\sec^2 \theta - 9}}{3 \sec \theta} \cdot 3 \sec \theta \tan \theta \, d\theta \]

\[ = \int \sqrt{\sec^2 \theta - 9} \tan \theta \, d\theta \]

\[ = \int \sqrt{\sec^2 \theta - 1} \tan \theta \, d\theta \]

\[ = \int \tan \theta \, d\theta \]

\[ = \int 3 \tan \theta \cdot \tan \theta \, d\theta \]

\[ = 3 \int \tan^2 \theta \, d\theta \]

\[ = 3 \int (\sec^2 \theta - 1) \, d\theta \]

\[ = 3 \left[ \tan \theta - \theta \right] + C \]

Now refer to the original substitution and the triangle above

\[ = 3 \left[ \frac{\sqrt{x^2 - 9}}{3} - \arccos \left( \frac{x}{3} \right) \right] + C \]

\[ = \sqrt{x^2 - 9} - 3 \arccos \left( \frac{x}{3} \right) + C \]
(c) $\int \frac{x^3}{x^2 - 1} \, dx$

**Method #1**  \( u \)-sub

Let \( u = x^2 - 1 \)  \( \Rightarrow \)  \( x^2 = u + 1 \)

\[ du = 2x \, dx \quad \Rightarrow \quad \frac{1}{2} \, du = x \, dx \]

\[
\int \frac{x^3}{x^2 - 1} \, dx = \int \frac{x \cdot x^2}{x^2 - 1} \, dx = \frac{1}{2} \int \frac{u + 1}{u} \, du
\]

\[
= \frac{1}{2} \int \left( 1 + \frac{1}{u} \right) \, du
\]

\[
= \frac{1}{2} \left[ u + \ln|u| \right] + C
\]

\[
= \frac{1}{2} (x^2 - 1) + \frac{1}{2} \ln|x^2 - 1| + C
\]

**Method #2** Long Division

\[
\frac{x^3}{x^2 - 1} \overline{x^2 - 1} \]

\[
\Rightarrow \quad \frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1}
\]

\[
\int \frac{x^3}{x^2 - 1} \, dx = \int x \, dx + \int \frac{x}{x^2 - 1} \, dx
\]

Use either \( u \)-sub or partial fractions

Let \( u = x^2 - 1 \)

\[ du = 2x \, dx \quad \Rightarrow \quad \frac{1}{2} \, du = x \, dx \]

\[
= \int x \, dx + \frac{1}{2} \int \frac{1}{u} \, du
\]

\[
= \frac{x^2}{2} + \frac{1}{2} \ln|u| + C
\]

\[
= \frac{x^2}{2} + \frac{1}{2} \ln|x^2 - 1| + C'
\]

Note the answers from the two methods look different but they are both correct, they just differ by a constant. (Recall: any two antiderivatives of a given function differ by a constant.)
4. (10 pts.) Find the interval of convergence for the series \( \sum_{k=1}^{\infty} \frac{3}{k5^k} x^k \).

Apply the Ratio test:

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{3x^{k+1}/(k+1)5^{k+1}}{3x^k/k5^k} \right| = \lim_{k \to \infty} \left| \frac{3x}{(k+1)5} \cdot \frac{k5^k}{3x^k} \right| = \lim_{k \to \infty} \left| \frac{x^{k+1}}{x^k} \cdot \frac{5}{5k+1} \cdot \frac{k}{k+1} \right|
\]

\[
= \lim_{k \to \infty} \left| \frac{x}{5} \cdot \frac{k}{k+1} \right| = \left| \frac{x}{5} \right| \quad \text{since} \quad \frac{k}{k+1} \to 1 \quad \text{as} \quad k \to \infty \quad \text{by l'Hospital's rule}
\]

The Ratio test tells us that a series converges if \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1 \)

\( \Rightarrow \) this power series converges if \( \left| \frac{x}{5} \right| < 1 \) \( \iff \) \( |x| < 5 \) \quad \text{Radius of convergence is 5}

\( \iff -5 < x < 5 \quad \text{need to check endpts. on this interval of convergence}

Check endpts. for convergence:

If \( x = -5 \)

\[
\sum_{k=1}^{\infty} \frac{3}{k5^k} (-5)^k = \sum_{k=1}^{\infty} \frac{3}{k} \left( \frac{-5}{5} \right)^k = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{3}{k} = 3 \sum_{k=1}^{\infty} \left( \frac{-1}{5} \right)^k
\]

This is an Alternating Harmonic series which we know converges by the AST.

If \( x = 5 \)

\[
\sum_{k=1}^{\infty} \frac{3}{k5^k} (5)^k = \sum_{k=1}^{\infty} \frac{3}{k} (\frac{5}{5})^k = \sum_{k=1}^{\infty} \frac{3}{k} \frac{1}{k} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2}
\]

This is the Harmonic series which we know diverges.

Therefore, \( \sum_{k=1}^{\infty} \frac{3}{k5^k} x^k \) converges for \( -5 \leq x < 5 \)

5. (10 pts.) Determine if \( \sum_{k=2}^{\infty} \frac{1}{k \ln k} \) converges or diverges. Justify your answer.

\( f(x) = \frac{1}{x \ln x} \) is continuous, positive, and decreasing for \( x \geq 2 \) so we can use the Integral test.

\[
\int_{2}^{\infty} \frac{1}{x \ln x} \, dx = \lim_{b \to \infty} \frac{\ln x}{\ln x} \left|_{2}^{b} \right. \left. \frac{1}{x} \, dx = \ln x \right|_{2}^{b} \frac{1}{x} \, dx
\]

\[
= \lim_{b \to \infty} \left[ \frac{\ln b}{\ln 2} - \frac{1}{\ln 2} \right]
\]

\[
= \lim_{b \to \infty} \left( \frac{\ln b}{\ln 2} \right) + \frac{1}{\ln 2} = \frac{1}{\ln 2}
\]

\( \Rightarrow \) \( \int_{2}^{\infty} \frac{1}{x \ln x} \, dx \) converges

Therefore, \( \sum_{k=2}^{\infty} \frac{1}{k \ln k} \) converges by the Integral test.
6. (10 pts.) Use separation of variables to solve the following initial value problem. Be sure to use the initial condition to determine the value of any constant you introduce.

\[
\frac{dy}{dt} = t^2 e^y \quad \text{with} \quad y(3) = 0
\]

\[
\frac{dy}{e^y} = t^2 \, dt \Rightarrow e^{-y} \, dy = t^2 \, dt
\]

Integrate both sides

\[
\int e^{-y} \, dy = \int t^2 \, dt \quad \Rightarrow \quad -e^{-y} = \frac{t^3}{3} + C \quad \iff \quad e^{-y} = -\frac{t^3}{3} - C
\]

Use initial value \(y(3) = 0\) to find \(C\) : plug in \(x = 3\), \(y = 0\)

\[
e^0 = -\frac{3^3}{3} - C \quad \iff \quad 1 = -9 - C \quad \iff \quad C = -10
\]

Thus,

\[
e^{-y} = -\frac{3^3}{3} - (-10) \quad \Rightarrow \quad e^{-y} = 10 - \frac{t^3}{3} \quad \Rightarrow \quad e^{-y} = \frac{30 - t^3}{3}
\]

Take the natural log to solve for \(y\)

\[
y = -\ln \left( \frac{30 - t^3}{3} \right) \quad \Rightarrow \quad y = -\ln \left( \frac{30 - t^3}{3} \right)
\]

7. (8 pts) Consider the region bounded by \(y = 16 - x^2\) and the \(x\)-axis. Set up, but do not evaluate, the integral that represents the volume of the solid found by revolving this region about the line \(y = -3\).

Integrate in terms of \(x\)

inner radius \(r = 3\)

outer radius \(R = 3 + y = 3 + (16 - x^2) = 19 - x^2\)

Volume of Solid

\[
V = \int_{-4}^{4} \left( \pi R^2 - \pi r^2 \right) \, dx
\]

\[
= \pi \int_{-4}^{4} \left[ (19 - x^2)^2 - 9 \right] \, dx
\]

\[
\text{Find } x\text{-intercepts for limits of integration}
\]

\[
y = 0 \quad \text{where the graph of } y = 16 - x^2 \text{ crosses the } x\text{-axis}
\]

\[
0 = 16 - x^2 \quad \Rightarrow \quad x^2 = 16 \quad \Rightarrow \quad x = \pm 4
\]

Volume of Washer

\[
\text{Volume of Washer} \approx \left( \pi R^2 - \pi r^2 \right) \Delta x
\]
8. (3 pts. each) Complete the following.

(a) The series $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots$ converges to $\frac{1}{e}$.

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots
\]

\[
\Rightarrow \frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots
\]

(b) If $f$ is decreasing, but concave up, then when a midpoint sum is used to approximate $\int_a^b f(x) \, dx$ the approximation will be underestimated because a rectangle whose height is determined by the value of $f$ at the midpoint of a subinterval has the same area as a trapezoid formed by using the tangent line to $f$ at the midpoint of the same subinterval. Since the function is concave up, the tangent line sits below the curve so the trapezoid sits below the function. This means the area is an underestimate. 

(c) The $n^{th}$ term test allows us to conclude that a series of the form $\sum_{k=1}^{\infty} a_k$ will converge if …

\[
\lim_{k \to \infty} a_k = 0 \quad \text{[i.e., if the individual terms of the series do not go to zero]}
\]

… then the series diverges.

(d) The series $1 + x + x^2 + x^3 + x^4 + \cdots$ converges for all $x$ such that $|x| < 1$. For such an $x$, the series converges to $\frac{1}{1-x}$.

(e) The Alternating Series Test states that …

\[\text{if the terms of a series alternate in sign, decrease in magnitude, and approach zero, then the series converges}\]

(f) Error bounds for using the $n^{th}$ order Taylor Polynomial $P_n(x)$ to estimate $f(x)$ on an interval $I$ may be determined using: $|f(x) - P_n(x)| \leq \frac{K}{(n+1)!}|x - x_0|^{n+1}$, where $K$ is found using the $n+1$ derivative such that …

\[
|f^{(n+1)}(x)| \leq K
\]

\[\text{in other words, } K \text{ is greater than or equal to the maximum value of } |f^{(n+1)}(x)| \text{ on the interval } I.\]