Math 205A Final Exam (65 points)

Name: Solutions

- Check that you have 7 questions on four pages.
- Show all your work to receive full credit for a problem.

1. (7 points) Let $A$ be a $4 \times 3$ matrix, $B$ be a $3 \times 4$ matrix and $C$ be a $4 \times 4$ matrix, with $\det(AB) = 2$ and $\det C = -10$.

   (a) Find $\det ABC$. Is $ABC$ an invertible matrix? Explain.

   \[
   \det(ABC) = (\det AB)(\det C) = 2(-10) = -20
   \]

   \[
   \det(ABC) \neq 0
   \]

   So $ABC$ is an invertible matrix.

(b) Find $\det B^T A^T$. Explain.

\[
\det(B^T A^T) = (\det B^T)(\det A^T)
\]

\[
= (\det B)(\det A)
\]

But we don't know $\det B$ and $\det A$.

So let's try something else.

\[
\det(B^T A^T) = \det(AB)^T = \det(AB) = 2
\]
2. (12 points) Let \( \vec{y} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \) and \( \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \). Let \( W = \text{Span}\{\vec{u}_1, \vec{u}_2\} \).

(a) Is \( \{\vec{u}_1, \vec{u}_2\} \) an orthogonal basis for \( W \)? Explain.

\[ \vec{u}_1 \cdot \vec{u}_2 = 0 \cdot 1 - 1 = -1 \]

So \( \{\vec{u}_1, \vec{u}_2\} \) is an orthogonal set.

It is an orthogonal set of non-zero vectors and hence it is linearly independent.

It also spans \( W \).

Thus, it is an orthogonal basis for \( W \).

(b) Compute the distance from \( \vec{y} \) to \( W \).

\[
\hat{\vec{z}} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1 + \frac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} \vec{u}_2
\]

\[
= \frac{4 + 4 + 0}{4 + 1 + 1} \vec{u}_1 + \frac{0 + 4 + 0}{0 + 1 + 1} \vec{u}_2 = \frac{8}{6} \vec{u}_1 + \frac{4}{2} \vec{u}_2
\]

\[
= \frac{4}{3} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 4/3 \\ -2/3 \end{bmatrix}
\]

\[
\begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}
\]

\[
\vec{z} = \vec{y} - \hat{\vec{z}} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 8/3 \\ 4/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ 2/3 \end{bmatrix}
\]

Distance from \( \vec{y} \) to \( W \) is \( ||\vec{z}|| \)

\[
= \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{12}{27}} = \sqrt{\frac{4}{9}} = \frac{2}{3}
\]

or \( \frac{2}{\sqrt{3}} \).
Problem 2 continued from previous page. In parts (c) and (d) below, \( W \) and \( \overrightarrow{y} \) are the same as on the previous page.

(c) Suppose \( \overrightarrow{v} \) is a vector in \( \mathbb{R}^3 \) such that the distance between \( \overrightarrow{y} \) and \( \overrightarrow{v} \) is 1. Can \( \overrightarrow{v} \) be in \( W \)? Explain.

Distance between \( \overrightarrow{y} \) and \( W \nabla \overrightarrow{y} \) and \( \overrightarrow{u} \). So \( \| \overrightarrow{y} - \overrightarrow{u} \| = \frac{2}{\sqrt{3}} \) (from (b))

i.e \( \| \overrightarrow{y} - \overrightarrow{u} \| = 1.15 \)

Now \( \| \overrightarrow{y} - \overrightarrow{u} \| = 1 \). Thus \( \| \overrightarrow{y} - \overrightarrow{u} \| > \| \overrightarrow{y} - \overrightarrow{u} \| \). Since \( \overrightarrow{u} \) is the closest point in \( W \) to \( \overrightarrow{y} \), \( \overrightarrow{u} \) cannot be in \( W \).

(d) In your computation in part (b), did you find a vector in \( W^\perp \)? If so, what is that vector?

In part (b), \( \overrightarrow{z} \) is a vector in \( W^\perp \).

Since \( \overrightarrow{z} \cdot \overrightarrow{u} = 0 \), \( \overrightarrow{z} \cdot \overrightarrow{u_2} = 0 \).

3. (6 points) \( H = \{ \text{all polynomials } \overrightarrow{p}(t) \text{ in } \mathbb{P}_2 \text{ such that } \overrightarrow{p}(0) = 0 \} \). Is \( H \) a subspace of \( \mathbb{P}_2 \)? Explain.

A polynomial in \( \mathbb{P}_2 \) is of the form

\[ \overrightarrow{p}(t) = a_0 + a_1 t + a_2 t^2. \]

Since \( \overrightarrow{p}(0) = 0 \), \( a_0 + 0 + 0 = 0 \); i.e., \( a_0 = 0 \).

So a polynomial in \( H \) is of the form

\[ \overrightarrow{p}(t) = a_1 t + a_2 t^2. \]

Thus \( \overrightarrow{p}(t) \) is a linear combination of \( t \) and \( t^2 \).

So \( H = \text{Span } \{ t, t^2 \} \).

Hence \( H \) is a subspace of \( \mathbb{P}_2 \).
4. (12 points) Define a linear transformation $T : \mathbb{M}_{2 \times 2} \to \mathbb{R}^2$ by

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a+d \\ b+c \end{bmatrix}.$$ 

(a) Find a set of matrices that spans the kernel (or null space) of $T$.

$$T \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} a \quad d \\ b \quad c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

So $a+d=0$, $b+c=0$. Thus, $a=-d$, $b=-c$. 

Thus, every matrix in kernel of $T$ is of the form $\begin{bmatrix} -d & 0 \\ c & -d \end{bmatrix}$, which can be written in terms of the basis matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus, kernel of $T = \text{Span} \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \}$.

(b) The kernel of $T$ is a subspace of $\mathbb{M}_{2 \times 2}$. Use the spanning set you found in part (a) to find a basis for the kernel of $T$.

$\mathbb{M}_{2 \times 2}$ is isomorphic to $\mathbb{R}^4$ and under the standard isomorphism, the spanning set in part (a) corresponds to the set $\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \}$ in $\mathbb{R}^4$.

Thus, the spanning set in part (a) is a basis for the kernel of $T$.

(c) Is $T$ one-to-one? Explain.

Since there are infinitely many vectors in the kernel of $T$ (as seen in part (a)), $T$ is not 1-1 because more than one vector is mapped to the zero vector under $T$. 


Problem 4 continued from previous page. In part (d) below, the linear transformation $T$ is the same as on the previous page.

(d) Is \[
\begin{bmatrix}
8 \\
-5
\end{bmatrix}
\] in the range of $T$? If so, find a matrix $A$ such that $T(A) = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$. If not, explain why not.

\[T(A) = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} a+d \\ b+c \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \end{bmatrix} \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

So $a+d = 8$, $a = 2$, $d = 6$, $b = -1$, $c = -4$ satisfy these eqns.

$b+c = -5$ Thus $A = \begin{bmatrix} 2 & -1 \\ -4 & 6 \end{bmatrix}$ is such that $T(A) = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

So \[
\begin{bmatrix} 8 \\ -5 \end{bmatrix}
\] is in the range of $T$.

5. (6 points) Let $\overline{p}_1(t) = 2-t$ and $\overline{p}_2(t) = 7t$ be polynomials in $P_1$.

(a) Verify that $B = \{\overline{p}_1, \overline{p}_2\}$ is a basis for $P_1$ by showing that the set satisfies the two conditions in the definition of a basis.

$P_1$ is isomorphic to $\mathbb{R}^2$ and we will use the isomorphism $a_1 + a_2 t \mapsto \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. So $\overline{p}_1(t) \mapsto \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\overline{p}_2(t) \mapsto \begin{bmatrix} 7 \\ 0 \end{bmatrix}$.

Check if $\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}\}$ is a basis for $\mathbb{R}^2$:

\[
\begin{bmatrix} 2 & 7 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

Pivot in every column means the set is linearly ind.

Pivot in every row means the set spans $\mathbb{R}^2$.

Thus the set $\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

Hence $B = \{\overline{p}_1, \overline{p}_2\}$ is a basis for $P_1$.

(b) Find the polynomial $\overline{q}$ such that $[\overline{q}]_B = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

So $\overline{q} = 5 \overline{p}_1 + (-1) \overline{p}_2$.

$\overline{q}(t) = 5(2-t) - 1(7t)$
$= 10 - 5t - 7t$
$= 10 - 12t$.
6. (12 points) A $3 \times 3$ matrix $A$ has only two eigenvalues, $-1$ and $5$. A basis for the eigenspace corresponding to $-1$ is \[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\] and a basis for the eigenspace corresponding to $5$ is \[
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}
\].

Let \[ \vec{x} = \begin{bmatrix}
-2 \\
-2 \\
-2
\end{bmatrix} \]

(a) Find $A\vec{x}$.

\[ \vec{x} = \begin{bmatrix}
-2 \\
-2 \\
-2
\end{bmatrix} = -2 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \]. So $\vec{x}$ is in the eigenspace corresponding to $-1$ because it is a multiple of the basis vector. Hence $\vec{x}$ is an eigenvector of $A$ with the corresponding eigenvalue $-1$.

So $A\vec{x} = -1 \cdot \vec{x} = \begin{bmatrix}
2 \\
2 \\
2
\end{bmatrix}$

(b) What is dimension of Nul $A + I$? What is rank of $A + I$? Explain.

\[ \dim \text{ Nul} (A + I) = \text{dimension of the eigenspace corresponding to } -1 \]

\[ = 1 \]

Number of columns of $A + I = \dim \text{ Nul} (A + I) + \text{rank} (A + I)$

\[ 3 = 1 + \text{rank} (A + I) \]

So $\text{rank} (A + I) = 2$. 
Problem 6 continued from previous page. In parts (c) and (d) below, the matrix \(A\) is the same as on the previous page.

(c) Are the columns of the matrix \(A - 5I\) linearly independent? Explain.

5 is an eigenvalue of \(A\).
So the eqn. \((A - 5I)x = 0\) has infinitely many solns.
Hence the RREF of \(A - 5I\) does not have a pivot in every column because the eqn. \((A - 5I)x = 0\) has free variables.
Hence the columns of \(A - 5I\) are not lin. ind.

(d) Is \(A\) diagonalizable? Explain.

\(A\) has only two eigenvalues, -1 and 5.
Sum of the dimensions of the corresponding eigenspaces = 1 + 1 = 2.
Thus the sum ≠ 3.
Hence \(A\) is not diagonalizable.
7. (10 points) The length of a spring changes when we apply a force to it. Hooke's law tells us that the force \( f \) and the length \( l \) are related by the equation \( l = a + bf \) where \( a \) and \( b \) are constants that depend on the spring. You would like to find these constants for a particular spring. To this end you collect the following experimental data by suspending a weight (which gives the force \( f \) in ounces) from the spring and then measuring the length \( l \) (in inches) of the spring.

<table>
<thead>
<tr>
<th>( f )</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l )</td>
<td>8.2</td>
<td>11.6</td>
<td>14.3</td>
<td>17.5</td>
</tr>
</tbody>
</table>

Find \( a \) and \( b \) so that the Hooke's law equation is a least-squares fit to the data. (Start by using the given data to write a system of linear equations to determine \( a \) and \( b \).)

System of eqns: \[
\begin{align*}
& a + 2b = 8.2 \\
& a + 4b = 11.6 \\
& a + 6b = 14.3 \\
& a + 8b = 17.5
\end{align*}
\]

Let \( A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \\ 1 & 8 \end{bmatrix} \), \( \overline{x} = \begin{bmatrix} a \\ b \end{bmatrix} \), \( \overline{l} = \begin{bmatrix} 8.2 \\ 11.6 \\ 14.3 \\ 17.5 \end{bmatrix} \). Then the system can be written as \( A \overline{x} = \overline{l} \).

Normal eqns: \( A^T A \overline{x} = A^T \overline{l} \)

\[
A^T A = \begin{bmatrix} 1 & 2 & 4 & 6 & 8 \\ 2 & 4 & 6 & 8 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 6 & 8 \\ 8 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 14 & 14 & 14 \\ 14 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \\ 20 & 20 & 20 & 20 \end{bmatrix}
\]

\( A^T \overline{l} = \begin{bmatrix} 8.2 \\ 11.6 \\ 14.3 \\ 17.5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 6 & 8 \\ 2 & 4 & 6 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 8.2 + 11.6 + 14.3 + 17.5 \\ 2(8.2) + 4(11.6) + 6(14.3) + 8(17.5) \\ 2(8.2) + 4(11.6) + 6(14.3) + 8(17.5) \\ 2(8.2) + 4(11.6) + 6(14.3) + 8(17.5) \end{bmatrix} = \begin{bmatrix} 57.6 \\ 288.6 \end{bmatrix} \)

Augmented matrix for \( A^T A \overline{x} = A^T \overline{l} \) is \[
\begin{bmatrix}
4 & 20 & 51.6 & 0 & 5.25 \\
20 & 120 & 288.6 & 0 & 1.53
\end{bmatrix}
\]

So \( a = 5.25 \), \( b = 1.53 \).