Math 106: Review for Final Exam, Part I - SOLUTIONS

1. Find the following. [See Review for Exam II for integration tips and strategies.]

(a) Let \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( du/3 = x^2 \, dx \).

\[
\int 12x^2 \cos(x^3) \, dx = 12 \int \cos(x^3) \, x^2 \, dx
\]

\[
= 12 \int \cos(u) \, \frac{du}{3}
\]

\[
= 4 \sin(u) + C
\]

\[
= 4 \sin(x^3) + C
\]

(b) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-3x} \Rightarrow v = \frac{e^{-3x}}{-3} \).

\[
\int_0^\infty xe^{-3x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-3x} \, dx
\]

\[
= \lim_{t \to \infty} \left[ \frac{e^{-3x} \, x}{-3} \bigg|_0^t - \int_0^t e^{-3x} \, dx \right]
\]

\[
= \lim_{t \to \infty} \left[ \frac{-x}{3e^{3t}} - \frac{1}{9} \bigg|_0^t \right]
\]

\[
= \lim_{t \to \infty} \left[ \frac{-t}{3e^{3t}} - \frac{1}{9e^3} \right] - \left[ \frac{0}{3e^0} - \frac{1}{9e^0} \right]
\]

\[
= (0 - 0) - (0 - 1/9)
\]

\[
= 1/9
\]

So, the integral converges (to this value).

(c) This integral is improper at \( x = 4 \) because the integrand has a vertical asymptote there, so we split into two integrals.

\[
\int_0^6 \frac{dx}{(x-4)^2} = \int_0^4 \frac{dx}{(x-4)^2} + \int_4^6 \frac{dx}{(x-4)^2}
\]

\[
= \lim_{a \to 4^-} \int_0^a \frac{dx}{(x-4)^2} + \lim_{b \to 4^+} \int_0^6 \frac{dx}{(x-4)^2}
\]

\[
= \lim_{a \to 4^-} \left[ -\frac{1}{x-4} \right]_0^a + \lim_{b \to 4^+} \left[ -\frac{1}{x-4} \right]_b^6 
\]

\[
= \left. -\frac{1}{x-4} \right|_0^a + \left. -\frac{1}{x-4} \right|_b^6
\]

\[
= \left. \frac{-1}{a-4} \right|_0^a + \left. \frac{-1}{b-4} \right|_b^6
\]

\[
= \lim_{a \to 4^-} \left[ \frac{-1}{a-4} - \frac{-1}{0-4} \right] + \lim_{b \to 4^+} \left[ \frac{-1}{6-4} - \frac{-1}{b-4} \right]
\]

Since \( \lim_{a \to 4^-} \frac{-1}{a-4} = \infty \) and \( \lim_{b \to 4^+} \frac{-1}{b-4} = \infty \), this integral diverges (to \( \infty \)).

(d) Partial Fractions:

Write \( \frac{3x^2 + 2x - 5}{(x^2 + 1)(x-4)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x-4} \). Now multiply both sides by \((x^2 + 1)(x-4)\) to get

\[
3x^2 + 2x - 5 = (Ax + B)(x-4) + C(x^2 + 1).
\]

Let \( x = 4 \). Then \( 51 = C(17) \), so \( C = 3 \).

Let \( x = 0 \). Then \( -5 = B(-4) + 3(1) \), so \( B = 2 \).
Let $x = 1$. Then $0 = (A(1) + 2)(-3) + 3(2)$, so $A = 0$.

\[
\int \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} \, dx = \int \left[ \frac{2}{x^2 + 1} + \frac{3}{x - 4} \right] \, dx \\
= 2 \arctan x + 3 \ln |x - 4| + D
\]

(e) Let $u = \sec x$, so $du = \sec x \tan x \, dx$.

New limits: $x = 0 \Rightarrow u = \sec 0 = 1/\cos 0 = 1$ and $x = \pi/3 \Rightarrow u = \sec(\pi/3) = 1/\cos(\pi/3) = 2$.

\[
\int_0^{\pi/3} \tan^3 x \sec^6 x \, dx = \int_0^{\pi/3} \tan^2 x \sec^4 x \sec x \tan x \, dx \quad \text{Break off a } \sec x \tan x.
\]

\[
= \int_0^{\pi/3} (\sec^2 x - 1) \sec^4 x \sec x \tan x \, dx \quad \text{Use } \tan^2 x = \sec^2 x - 1.
\]

\[
= \int_1^2 (u^2 - 1)u^4 \, du \quad \text{Change the limits. See above.}
\]

\[
= \int_1^2 (u^6 - u^4) \, du
\]

\[
= \left[ \frac{u^7}{7} - \frac{u^5}{5} \right]_1^2
\]

\[
= \frac{2^7}{7} - \frac{2^5}{5} - \left[ \frac{1^7}{7} - \frac{1^5}{5} \right]
\]

\[
= \frac{418}{35}
\]

This is about 11.943.

(f) Let $x = 5 \sin t$, so $dx = 5 \cos t \, dt$.

\[
x^2 + y^2 = 5^2 \Rightarrow y = \sqrt{25 - x^2}
\]

\[
\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5)
\]

\[
\cos t = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5 \cos t = \sqrt{25 - x^2}
\]

\[
\int \sqrt{25 - x^2} \, dx = \int 5 \cos t \cdot 5 \cos t \, dt \quad \text{Use } dx \text{ and } \cos t \text{ from above.}
\]

\[
= \int 25 \cos^2 t \, dt
\]

\[
= 25 \left[ \frac{1}{2} + \frac{\cos(2t)}{2} \right] \, dt \quad \text{Use } \cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2} \text{ or table #42.}
\]

\[
= 25 \left[ \frac{t}{2} + \frac{\sin(2t)}{4} \right] + C
\]

Let $u = 2t$ to integrate $\cos(2t)$.

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{2 \sin t \cos t}{4} \right] + C \quad \text{Use } \sin(2t) = \sin t \cos t \text{ and } x \text{ from above.}
\]

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{1}{2} \cdot \frac{x \sqrt{25 - x^2}}{5} \right] + C
\]

Use $\sin t$ and $\cos t$ from above.

\[
= 25 \left[ \frac{\arcsin(x/5)}{2} + \frac{x \sqrt{25 - x^2}}{250} \right] + C
\]
2. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_{-2}^{0} f(x) \, dx \) given the data in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

\[ L_4 = (2 + 3 + 6 + 10)(0.5) = 10.5 \quad R_4 = (3 + 6 + 10 + 11)(0.5) = 15 \quad T_4 = 0.5(L_4 + R_4) = 12.75 \]

We cannot compute \( M_4 \), which would require the values of \( f \) at \( x = -1.75, -1.25, -0.75, \) and \(-0.25 \). Instead, we find

\[ M_2 : M_2 = (3 + 10)(1) = 13. \]

Finally, \( S_4 = \frac{2M_2 + T_2}{3} = \frac{2(13) + 12.5}{3} = 7.7 \]

3. If you use numerical integration to estimate \( \int_{a}^{b} \ln x \, dx \), how would the following be ordered from least to greatest? \( L_{100}, R_{100}, M_{100}, T_{100}, S_{200} \).

The integrand is increasing and concave down, so we have \( L_{100} < T_{100} < S_{200} < M_{100} < R_{100} \).

What can you say with certainty about where \( \int_{a}^{b} \ln x \, dx \) would fit into your ordering?

It would fall somewhere between \( T_{100} \) and \( M_{100} \).

4. Find bounds for each of the following errors if \( I = \int_{0}^{2} e^{-5x} \, dx \).

(a) \( |I - R_{100}| \leq \frac{K_1(b - a)^2}{2n} = \frac{5(2 - 0)^2}{2(100)} = \frac{1}{10} \)

\( K_1 = \text{max of } |f'(x)| \text{ on } [0, 2] = \text{max of } 5e^{-5x} \text{ on } [0, 2] = 5 \) (occurs at \( x = 0 \))

(b) \( |I - T_{100}| \leq \frac{K_2(b - a)^3}{12n^2} = \frac{25(2 - 0)^3}{12(100)^2} = \frac{1}{600} \)

\( K_2 = \text{max of } |f''(x)| \text{ on } [0, 2] = \text{max of } 25e^{-5x} \text{ on } [0, 2] = 25 \) (occurs at \( x = 0 \))

(c) \( |I - M_{100}| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{25(2 - 0)^3}{24(100)^2} = \frac{1}{1200} \)

\( K_2 = \text{same as in previous part} \)

5. If \( I = \int_{0}^{2} e^{-5x} \, dx \), how many subdivisions are required to obtain a midpoint sum approximation with error of at most \( 1/1,000,000 \)?

From part (c) above, we know that \( |I - M_n| \leq \frac{K_2(b - a)^3}{24n^2} = \frac{25(2 - 0)^3}{24n^2} = \frac{25}{3n^2} \)

Thus, we want \( \frac{25}{3n^2} \leq \frac{1}{1,000,000} \), which is equivalent to \( \frac{25,000,000}{3} \leq n^2 \).

Taking the square root of each side results in \( \sqrt{\frac{25,000,000}{3}} \leq n \).

Since \( \sqrt{25,000,000/3} = 2886.751... \), we must at least 2887 subdivisions.
6. Use Euler’s Method with 3 steps to estimate \( y(3/4) \) if \( \frac{dy}{dx} = y - 3 \) and \( y(0) = 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( \frac{dy}{dx} \cdot \Delta x = \Delta y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>((-2)(0.25) = -0.5)</td>
</tr>
<tr>
<td>0.25</td>
<td>0.5</td>
<td>((-2.5)(0.25) = -0.625)</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.125</td>
<td>((-3.125)(0.25) = -0.78125)</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.90625</td>
<td>(-0.90625)</td>
</tr>
</tbody>
</table>

7. Write an integral equal to the area between \( y = 2x + 3 \) and \( y = x^2 + 7x - 3 \).

First, find where the curves intersect.

\[
x^2 + 7x - 3 = 2x + 3 \\
x^2 + 5x - 6 = 0 \\
(x + 6)(x - 1) = 0 \\
\Rightarrow x = -6, x = 1
\]

Between \( x = -6 \) and \( x = 1 \), \( y = 2x + 3 \) is above \( y = x^2 + 7x - 3 \). (Plug in \( x = 0 \) to check.) So, the area between them is

\[
\int_{-6}^{1} [(2x + 3) - (x^2 + 7x - 3)] \, dx \\
\text{This equals } 343/6.
\]

8. Compute the arc length of \( y = \sqrt{1 - x^2} \) from \( x = 0 \) to \( x = 1/2 \).

First, we find \( f'(x) = \frac{1}{2} (1 - x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{1 - x^2}} \).

Thus, \( [f'(x)]^2 = \frac{x^2}{1 - x^2} \).

\[
\int_{0}^{\frac{1}{2}} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{\frac{1}{2}} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx \\
\text{This is the definition of arc length.}
\]

\[
= \int_{0}^{\frac{1}{2}} \sqrt{\frac{1}{1 - x^2} + \frac{x^2}{1 - x^2}} \, dx \\
= \int_{0}^{\frac{1}{2}} \sqrt{\frac{1}{1 - x^2}} \, dx \\
= \int_{0}^{\frac{1}{2}} \frac{\sqrt{1}}{\sqrt{1 - x^2}} \, dx \\
= \arcsin x \bigg|_{0}^{\frac{1}{2}} \\
= \arcsin(1/2) - \arcsin(0) \\
= \frac{\pi}{6} - 0 \\
= \frac{\pi}{6}
\]

9. Consider the region bounded by \( y = 0, x = 2, \) and \( y = x^2 \). Write an integral equal to the volume of the object created when the region is revolved about

(a) the \( x \)-axis

Slice vertically into disks.
volume of slice \( \approx \pi r^2 \Delta x \)
\[ = \pi y^2 \Delta x \]
\[ = \pi (x^2)^2 \Delta x \]
\[ = \pi x^4 \Delta x \]

\[
\text{total volume} = \pi \int_0^2 x^4 \, dx
\]

(b) **the line** \( x = 5 \)

Slice horizontally into washers.

volume of slice \( \approx \pi R^2 \Delta y - \pi r^2 \Delta y \)
\[ = \pi (5 - x)^2 \Delta y - \pi (3)^2 \Delta y \]
\[ = \pi [(5 - \sqrt{y})^2 - 3^2] \Delta y \]

\[
\text{total volume} = \pi \int_0^4 [(5 - \sqrt{y})^2 - 3^2] \, dy
\]

10. **Find the solution to** \( \frac{dy}{dx} = \cos \frac{x}{y^2} \) **that passes through** \((0, 2)\). **Use separation of variables.**

\[
\int y^2 \, dy = \int \cos x \, dx
\]
\[ y^3 / 3 = \sin x + C \]
\[ y^3 = 3 \sin x + D \]
\[ y = \sqrt{3 \sin x + D} \]

When \( x = 0 \), we have \( y = 2 \), so \( 2 = \sqrt{3 \sin 0 + D} \), or \( 2 = \sqrt{D} \). Thus, \( D = 8 \).

Therefore, the solution is \( y = \sqrt{3 \sin x + 8} \).

11. **The probability density function (pdf) of the weights of newborn toads in a certain pond is given by** \( f(x) = \frac{k}{(x + 1)^4} \), where \( x \) is the weight (in ounces). Note that the domain is \( x \geq 0 \) since no toad can have a negative weight.

(a) **What must be the value of** \( k \)?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_0^\infty \frac{k}{(x + 1)^4} \, dx = \lim_{t \to \infty} \int_0^t \frac{k}{(x + 1)^4} \, dx
\]
\[ = \lim_{t \to \infty} \left[ \frac{k(x + 1)^{-3}}{-3} \right]_0^t
\]
\[ = \lim_{t \to \infty} \left[ \frac{k}{-3} (t + 1)^3 \right]_0^t
\]
\[ = \lim_{t \to \infty} \frac{k}{-3} (t + 1)^3 - \frac{k}{-3(0 + 1)^3}
\]
\[ = 0 - \frac{k}{3}
\]
\[ = \frac{k}{3}
\]
So, we have $k/3 = 1$ or $k = 3$.

(b) **What fraction of the newborn toads weigh more than one ounce?**

\[
\int_1^\infty \frac{3}{(x+1)^4} \, dx = \lim_{t \to \infty} \int_1^t \frac{3}{(x+1)^4} \, dx \\
= \lim_{t \to \infty} \left[ \frac{3}{(x+1)^3} \right]_1^t \\
= \lim_{t \to \infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \\
= 0 - \frac{1}{8} \\
= \frac{1}{8}
\]

Note that we could instead have computed $1 - \int_0^1 \frac{3}{(x+1)^4} \, dx$ and gotten the same answer.