Mathematics 205  
Final Exam  
April 10, 2012

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You must show all work to receive credit.

No electronic devices other than calculators are permitted.

Give exact answers (such as $\ln 5$ or $e^2$) unless requested otherwise.
1. Let \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \), and \( \vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \).

(a) Are these vectors linearly independent? Explain.

(b) Are these any of vectors orthogonal? Explain.

(c) Let \( W = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \). Find an orthogonal basis for \( W \).
2. Recall the difference between $\sqrt{4}$ and solving $x^2 = 4$. The former is 2 while the latter has the solutions $x = 2$ or $x = -2$. We call that positive root of $x^2 - 4 = 0$ the principal root of 4 and write $\sqrt{4}$. We can extend the definition to matrices. By diagonalizing a matrix and taking the square roots of the eigenvalues, we may compute the square root of a matrix.

(a) Let $A = \begin{bmatrix} 9 & 15 \\ 0 & 4 \end{bmatrix}$. Compute $\sqrt{A}$.

(b) Write all solutions to $x^2 = A$.

(c) Suppose $A$ was a $3 \times 3$ matrix with 3 positive, distinct eigenvalues. How many solutions would there be to $x^2 = A$?
3. We will use the following three statements in the Invertible Matrix Theorem.

- $A$ is an invertible matrix.
- There is an $n \times n$ matrix $C$ such that $CA = I$.
- There is an $n \times n$ matrix $D$ such that $AD = I$.

(a) Show that if $AB$ is invertible then $A$ is invertible. You may not assume that $B$ is invertible to do this problem.

(b) Show that if $AB$ is invertible then $B$ is invertible. You may not assume that $A$ is invertible to do this problem.
4. Let \( \mathcal{K} \) be the set of \( 3 \times 3 \) skew-symmetric matrices \((A = -A^T)\) and let \( \mathcal{S} \) be the set of \( 3 \times 3 \) symmetric matrices \((A = A^T)\). Let \( T : \mathcal{K} \to \mathcal{S} \) be defined by \( T(A) = A^2 \).

(a) Verify that the square of a skew-symmetric matrix is a symmetric matrix so that the statement “\( T : \mathcal{K} \to \mathcal{S} \)” makes sense. Recall that this is read as “\( T \) is a map from skew-symmetric matrices to symmetric matrices.”

(b) Is \( T \) a linear map?

(c) What does it mean for a map to be onto? Is \( T \) onto? (Hint: think of the dimensions of the spaces involved.)

(d) What does it mean for a map to be one-to-one? Is \( T \) one-to-one?
5. Suppose \( B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & 0 & k \\ 2 & 5 & 6 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix} \).

(a) If \( k = 1 \), what is \( \det(B) \)?

(b) What value of \( k \) makes \( B \) not invertible?
6. Balance the following chemical reaction using techniques learned in class.

$$\text{PbN}_6 + \text{CrMn}_2\text{O}_8 \rightarrow \text{Pb}_3\text{O}_4 + \text{Cr}_2\text{O}_3 + \text{MnO}_2 + \text{NO}. $$
7. The problem deals with the vector space of \( n \times n \) matrices \( \mathcal{M}_{n \times n} \).

(a) Explain why \( \dim \mathcal{M}_{n \times n} = n^2 \).

(b) Let \( A \in \mathcal{M}_{n \times n} \). Show that there are scalars \( c_0, c_1, c_2, \ldots, c_{n^2} \), not all 0, so that \( c_0 I_n + c_1 A + c_2 A^2 + \ldots + c_{n^2} A^{n^2} = O \). That is, there is a nonzero polynomial \( p \) of degree at most \( n^2 \) so that \( p(A) = O \) (where \( O \) is the \( n \times n \) zero matrix).