1. Consider the function \( f(x) = x^6 - 2x^3 \) on the interval \([-2, 2]\).

(a) Find the \( x \)- and \( y \)-coordinates of any and all local extrema and classify each as a local maximum or local minimum.

\[
f'(x) = 6x^5 - 6x^2
\]

\[
0 = 6x^2(x^3 - 1)
\]

\[
\Rightarrow x = 0, 1
\]

\[
\begin{array}{c|c|c|c}
-2 \leq x < 0 & 0 < x < 1 & 1 < x \leq 2 \\
f & \text{negative} & \text{negative} & \text{positive} \\
\end{array}
\]

\[y\text{-values: } f(0) = 0, f(1) = -1\]

So, \( f \) has a local minimum at \((1, -1)\); \((0, 0)\) is not a local extremum.

(b) Find the \( x \)- and \( y \)-coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the \( y \)-values at the local extrema and the endpoints.

\[y\text{-values: } f(-2) = 80, f(1) = -1, f(2) = 48\]

So, \( f \) has a global minimum at \((1, -1)\) and a global maximum at \((-2, 80)\).

(c) Find the \( x \)-coordinate(s) of any and all inflection points.

\[
f''(x) = 30x^4 - 12x
\]

\[
0 = 6x(5x^3 - 2)
\]

\[
\Rightarrow x = 0, \sqrt[3]{0.4}
\]

\[
\begin{array}{c|c|c|c}
x < 0 & 0 < x < \sqrt[3]{0.4} & \sqrt[3]{0.4} < x \\
f'' & \text{positive} & \text{negative} & \text{positive} \\
f'' & \text{concave up} & \text{concave down} & \text{concave up} \\
\end{array}
\]

So, the \( x \)-values of the inflection points of \( f \) are \( x = 0 \) and \( x = \sqrt[3]{0.4} \).

2. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is $9.00 per container, what dimensions will give the largest volume? [Students in the 1:10 section may omit this problem.]

area of circle = \( \pi r^2 \)    lateral area of cylinder = \( 2\pi rh \)    volume of cylinder = \( \pi r^2 h \)

Objective function: volume = \( V = \pi r^2 h \)

We need to get this down to a function of just one variable, so we use the

constraint equation: cost = 900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh

\[
900 = 6\pi r^2 + 10\pi rh
\]

\[
900 - 6\pi r^2 = 10\pi rh
\]

\[
\frac{900 - 6\pi r^2}{10\pi r} = h
\]

Substituting this back into the objective function gives
\[ V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10}(900r - 6\pi r^3). \]

Now that we have \( V \) as a function of just one variable, we find its maximum.

\[
V'(x) = \frac{1}{10}(900 - 18\pi r^2) \\
0 = \frac{1}{10}(900 - 18\pi r^2) \\
\Rightarrow 18\pi r^2 = 900 \\
\Rightarrow r^2 = \frac{50}{\pi} \\
\Rightarrow r = \sqrt{\frac{50}{\pi}}
\]

Thus, we have in fact found the global maximum at \( r = \sqrt{\frac{50}{\pi}} \).

And \( h = \frac{900 - 6\pi r^2}{10\pi} = \ldots \text{much simplifying} \ldots = \sqrt{\frac{72}{\pi}} \approx 4.787 \) inches.

3. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

\[
\begin{align*}
0 < x &= \sqrt{\frac{50}{\pi}} & \sqrt{\frac{50}{\pi}} &< x \\
f' &= \text{positive} & \text{negative}
\end{align*}
\]

Thus, we have in fact found the global maximum at \( r = \sqrt{\frac{50}{\pi}} \).

4. Use the Intermediate Value Theorem to show that \( f(x) = x^3 - 2x - 1 \) has a root on \([1, 2]\).

**IVT:** If \( f \) is continuous on \([a, b]\) and \( y \) is a number between \( f(a) \) and \( f(b) \), then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y \).

For the function given above, \( f(1) = -2 \) and \( f(2) = 3 \). Since 0 is a number between -2 and 3, the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0 \); this \( c \) is the desired root.

5. What (if anything) does the Extreme Value Theorem say about \( f(x) = x^2 \) on each of the following intervals?

**EVT:** If \( f \) is continuous on \([a, b]\), then \( f \) has both a maximum and a minimum on \([a, b]\).
(a) $[1, 4]$

$f$ has a maximum and a minimum on $[1, 4]$

(b) $(1, 4)$

The EVT doesn’t apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

6. Find the value of the constant $c$ that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.

MVT: If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there is a number $c$ between $a$ and $b$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

For our function, we have $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$.

And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

7. Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

overestimate $= L_4 = (15 + 11 + 8 + 4)(2) = 76$

underestimate $= R_4 = (11 + 8 + 4 + 3)(2) = 52$

(b) Interpret the expression $\int_{2}^{6} r(t) \, dt$ in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.

8. Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.

Let $G(x) = \int_{0}^{x} f(t) \, dt$ and $H(x) = \int_{-3}^{x} f(t) \, dt$.

(a) Compute $G(2)$, $G(4)$, and $H(4)$.

$G(2)$ is the area under $f$ between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$.

Similarly, $G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 3 + \frac{\pi}{2}$. 
\( H(4) \) is the area under \( f \) between \( t = -3 \) and \( t = 4 \). Remember that area below the \( t \)-axis counts as negative.

\[
H(4) = - (2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1) + \frac{1}{2} \cdot 2 \cdot 1 + [\text{area under } f \text{ from } 0 \text{ to } 4, \text{ found above as } G(4)] \\
= - 2 + \left[ 3 + \frac{\pi}{2} \right] \\
= 1 + \frac{\pi}{2}
\]

(b) \textbf{Where is } \( G \text{ increasing? Where is } G \text{ decreasing?} \)

For parts (b), (c), and (d), recall that we learned in class that \( G' = f \).

\( G \) is increasing where \( f \) is positive: \((-1, 4)\). Note that \( G \) has a horizontal slope at \( x = 2 \) but since \( f \) is positive on each side of \( t = 2 \), we say \( G \) is increasing at \( x = 2 \).

\( G \) is decreasing where \( f \) is negative: \([-4, -1)\).

(c) \textbf{Where is } \( G \text{ concave up? Where is } G \text{ concave down?} \)

\( G \) is concave up where \( f \) is increasing: \((-2, 0) \cup (2, 3)\).

\( G \) is concave down where \( f \) is decreasing: \((1, 2) \cup (3, 4)\).

(d) \textbf{At what } \( x \)-value(s) \( \) does \( G \text{ have a local maximum? At what } x \)-value(s) \( \) does \( G \text{ have a local minimum?} \)

\( G \) has a local maximum where \( f \) changes from positive to negative: never.

\( G \) has a local minimum where \( f \) changes from negative to positive: \( x = -1 \).

(e) \textbf{Find a formula that relates } \( G \text{ and } H \).

From their definitions, \( H(x) = \int_{-3}^{0} f(t) \, dt + G(x) = -2 + G(x) \).

(f) \textbf{How would your answers to (b), (c), and (d) change if the questions were about } \( H \text{ instead of } G? \)

They would not change at all because \( H'(x) = G'(x) \).

9. (a) \textbf{Use sigma notation to express } \( L_{10} \text{ and } M_{10} \text{ as approximations to } \int_{20}^{60} \ln x \, dx. \)

We’re subdividing the interval into 10 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{10} = 4 \).

\[
L_{10} = \left[ f(20) + f(24) + f(28) + ... + f(52) + f(56) \right] \Delta x \\
= \left[ \ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56) \right] \cdot 4 \\
= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4
\]

\[
M_{10} = \left[ f(22) + f(26) + f(30) + ... + f(54) + f(58) \right] \Delta x \\
= \left[ \ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58) \right] \cdot 4 \\
= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4
\]

(b) \textbf{Draw a sketch that represents the sum } \( M_4 \).

Now we’re subdividing the interval into 4 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{4} = 10 \).

Note that the height of each rectangle is determined by the \( y \)-value of the curve at the \( \text{middle} \) \( x \)-value of the rectangle (that is, at \( x = 25, 35, 45, 55 \)).
10. Find the following.

(a) all antiderivatives of \( 1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5} \)

Any such antiderivative will take the form \( x + x^2 + \frac{x^4}{4} + \frac{4x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + C \).

Note that we have used the facts that \( \sqrt{x} = x^{1/2} \) and \( 1/x^5 = x^{-5} \).

(b) \( \int_{1}^{7} \frac{3}{x} \, dx = 3 \ln |x| \bigg|_{1}^{7} = 3 \ln 7 - 3 \ln 1 = 3 \ln 7 \)

(c) \( \int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi \) This integral represents the area of a semicircle of radius 2.

(d) \( \frac{d}{dx} \int_{1}^{x} \sin \sqrt{t} \, dt = \sin \sqrt{x} \) The derivative of the area function is the original function.