(15) I. Define any three of these terms. Use a complete, mathematically correct sentence for each definition.

binary operation
group
ring
field
polynomial

A.

B.

C.
(10) II. Give examples of the following:

A. An integral domain with 71 elements.

B. An infinite cyclic subgroup of \( \langle \mathbb{R}^* , \cdot \rangle \), the non-zero real numbers under multiplication.

C. A field with 5 elements.

D. Two non-isomorphic abelian groups, each with 75 elements.

E. An integral domain that is not a field.

F. A polynomial of degree 71 in \( \mathbb{Z}[x] \) that is irreducible in \( \mathbb{Z}[x] \).

G. A non-trivial homomorphism from \( \mathbb{Z}_9 \) to \( \mathbb{Z}_6 \).

H. A homomorphism from \( \mathbb{Z}_4 \) to \( \mathbb{Z}_8 \) whose kernel is \( \{0\} \).
III. Draw the lattice diagram of subgroups of $\mathbb{Z}_{27}$, the integers modulo 27, where the operation is addition modulo 27.

IV. Fill in the blanks:

A. A generator for $\mathbb{Z}_8 \times \mathbb{Z}_9$ different from $(1, 1)$ is _______________________ .

B. The order of $(1, 2, 3)(2, 3, 4, 5)(1, 2)$ in $S_5$ is _______________________.

C. The number of zeroes of $x^3 + x + 1$ in $\mathbb{Z}_3$ is ________________.

D. The number of left cosets of $\langle 5 \rangle$ in $\mathbb{Z}_{20}$ is ________________.

E. The number of elements in the group $S_7$ is ________________.

F. The order of the group $D_5$ of symmetries of the regular pentagon is ________________.

G. Express $(1, 5, 7, 3, 4, 2) \in S_7$ as a product of transpositions ________________.

H. The order of $2 + \langle 4 \rangle$ in the group $\mathbb{Z}_{12} / \langle 4 \rangle$ is ________________.

I. The index of $A_5$ in $S_5$ is ____________________________________.

J. The number of right cosets of $\langle 5 \rangle$ in $\mathbb{Z}_{10}$ is ________________.

K. The order of the group $(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle (2, 2) \rangle$ is ________________.

L. The units in the ring $\langle \mathbb{Z}_9, +, \cdot \rangle$ are ________________.
(5) V. Let \( \phi : R \to R' \) be a ring homomorphism, which means that \( \phi(a + b) = \phi(a) + \phi(b) \) and \( \phi(ab) = \phi(a) \phi(b) \) for all \( a, b \in R \).

Prove that if \( a \in \ker(\phi) \) and \( r \in R \), \( ra \in \ker(\phi) \).

(5) VI. Factor \( x^5 + x^3 + x + 1 \) into irreducible factors over \( \mathbb{Z}_2[x] \)
(10) VII. Let $G$ be a group and let $H$ be a subgroup of $G$. Define the relation $\sim$ on $G$ by rule $a \sim b$ if and only if $ab^{-1} \in H$. Prove that $\sim$ is an equivalence relation on $G$. 
VIII. The identity function, $id$, is one of four automorphisms of $\mathbb{Z}_8$. If the others are denoted $\phi_1, \phi_2, \text{ and } \phi_3$, complete this table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$id(x)$</th>
<th>$\phi_1(x)$</th>
<th>$\phi_2(x)$</th>
<th>$\phi_3(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The elements of this set of automorphisms of $\mathbb{Z}_8$ (denoted $\operatorname{Aut}(\mathbb{Z}_8)$) are permutations of the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Therefore $\operatorname{Aut}(\mathbb{Z}_8)$ is isomorphic to a group of permutations with 4 elements. Which of the two groups of order 4 is $\operatorname{Aut}(\mathbb{Z}_8)$?
IX. A Boolean ring $B$ is a ring with the property that $a^2 = a$ for every $a \in B$.

A. Prove that $a + a = 0$ for every $a \in B$. [Hint: Expand $(a + a)^2$.]

B. Prove that $ab + ba = 0$ for all $a, b \in B$. [Hint: Expand $(a + b)^2$.]

C. Prove that $B$ is commutative. [Hint: Use Part B.]
1. The real numbers are closed under subtraction.
2. The empty set is an example of a ring.
3. Every cyclic group has a generator.
4. The real numbers form an abelian group under addition.
5. The real numbers form a group under multiplication.
6. $\mathbb{Z}_5 \times \mathbb{Z}_{25}$ is a cyclic group.
7. The number of elements in any subgroup of a finite group $G$ divides the number of elements in $G$.
8. Every permutation can be expressed as a product of cycles.
9. $S_6$ has no cyclic subgroups.
10. The composition of two permutations of a set $A$ is always a permutation of $A$.
11. Every left coset of a subgroup of a group $G$ is also a subgroup of $G$.
12. Every abelian group of order 8 contains a cyclic subgroup of order 8.
13. Every finite group of prime order is cyclic.
14. If $H \leq S_5$ and $H$ has 24 elements, then $S_5 / H$ is abelian.
15. The function $\phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_{12}$ defined by $\phi(n) = n$ for all $n \in \mathbb{Z}_6$ is a homomorphism.
16. $x^5 + x^3 + x + 1$ is irreducible over $\mathbb{Z}_2[x]$.
17. Every subgroup of a group $G$ is also a coset of $G$.
18. For any two groups $G$ and $G'$ there is a homomorphism from $G$ to $G'$.
19. Every field is an integral domain.
20. Every cyclic group is finite.