1. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is $9.00 per container, what dimensions will give the largest volume?

area of circle = $\pi r^2$  
lateral area of cylinder = $2\pi rh$  
volume of cylinder = $\pi r^2 h$

Objective function: $V = \pi r^2 h$

We need to get this down to a function of just one variable, so we use the

cost equation: $900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh$

$900 = 6\pi r^2 + 10\pi rh$

$900 - 6\pi r^2 = 10\pi rh$

$\frac{900 - 6\pi r^2}{10\pi r} = h$

Substituting this back into the objective function gives

$V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10} (900r - 6\pi r^3)$.

Now that we have $V$ as a function of just one variable, we find its maximum.

$V'(x) = \frac{1}{10} (900 - 18\pi r^2)$

Since $V'(x)$ never fails to exist, we just need to solve $V'(x) = 0$.

$0 = \frac{1}{10} (900 - 18\pi r^2)$

$\Rightarrow 18\pi r^2 = 900$

$\Rightarrow r^2 = \frac{50}{\pi}$

$\Rightarrow r = \sqrt{\frac{50}{\pi}}$

Thus, we have in fact found the global maximum at $r = \sqrt{50/\pi}$.

And $h = \frac{900 - 6\pi r^2}{10\pi r} = \ldots$ much simplifying... = $\sqrt{\frac{72}{\pi}} \approx 4.787$ inches.

2. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?

We know $\frac{db}{dt}$, and we want to find $\frac{da}{dt}$. 

[Diagram: You (top) pulling on line to boat (bottom) with line = a, distance to you = b, height = 6 feet]
So, we write an equation that relates \( a \) and \( b \) and then differentiate implicitly with respect to time \( t \).

\[
a^2 + b^2 = c^2 \]
\[
2a \frac{da}{dt} + 0 = 2b \frac{db}{dt}
\]
\[
\frac{da}{dt} = \frac{b}{a} \frac{db}{dt}
\]

At the moment in question, \( b = 10, a = 8 \) (by the Pythagorean Theorem), and \( \frac{db}{dt} = -3 \).

So, \( \frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75 \) feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

3. Use the Intermediate Value Theorem to show that \( f(x) = x^3 - 2x - 1 \) has a root on \([1, 2]\).

IVT: If \( f \) is continuous on \([a, b]\) and \( y \) is a number between \( f(a) \) and \( f(b) \), then there is a number \( c \) between \( a \) and \( b \) such that \( f(c) = y \).

For the function given above, \( f(1) = -2 \) and \( f(2) = 3 \). Since 0 is a number between -2 and 3, the IVT says there is a number \( c \) between 1 and 2 such that \( f(c) = 0 \); this \( c \) is the desired root.

4. What (if anything) does the Extreme Value Theorem say about \( f(x) = x^2 \) on each of the following intervals?

EVT: If \( f \) is continuous on \([a, b]\), then \( f \) has both a maximum and a minimum on \([a, b]\).

(a) \([1, 4]\)

\( f \) has a maximum and a minimum on \([1, 4]\)

(b) \((1, 4)\)

The EVT doesn’t apply because \((1, 4)\) is not a closed interval since its endpoints are not included.

5. Find the value of the constant \( c \) that the Mean Value Theorem specifies for \( f(x) = x^3 + x \) on \([0, 3]\). [Students in the 1:10 section may omit this problem.]

MVT: If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is a number \( c \) between \( a \) and \( b \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

For our function, we have \( \frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10 \).

And \( f'(x) = 3x^2 + 1 \), so \( f'(c) = 3c^2 + 1 \).

So, we solve \( 3c^2 + 1 = 10 \), which means \( c = \sqrt{3} \). (The other solution, \( x = -\sqrt{3} \), is not in our interval \([0, 3]\).)

6. Water is leaking out of a tank at a decreasing rate \( r(t) \) as shown below.

<table>
<thead>
<tr>
<th>time (min)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>rate (gal/min)</td>
<td>15</td>
<td>11</td>
<td>8</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

\[
\text{overestimate} = L_4 = (15 + 11 + 8 + 4)(2) = 76
\]
\[
\text{underestimate} = R_4 = (11 + 8 + 4 + 3)(2) = 52
\]

(b) Interpret the expression \( \int_2^6 r(t) \, dt \) in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval \([2, 6]\) minutes.
7. Consider the graph of \( f(t) \) shown. It is made of straight lines and a semicircle.

\[
\begin{array}{c}
\text{graph of } f(t) \\
\end{array}
\]

Let \( G(x) = \int_{0}^{x} f(t) \, dt \) and \( H(x) = \int_{-3}^{x} f(t) \, dt \).

(a) Compute \( G(2) \), \( G(4) \), \( G(-4) \), and \( H(4) \).

First, \( G(2) = \int_{0}^{2} f(t) \, dt \) is the area under \( f \) between \( t = 0 \) and \( t = 2 \). This is a rectangle plus a triangle and has area \( 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3 \).

Similarly, \( G(4) = \int_{0}^{4} f(t) \, dt = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 3 + \frac{\pi}{2} \).

Now, remembering that area below the \( t \)-axis counts as negative and that \( \int_{b}^{a} f(t) \, dt = -\int_{a}^{b} f(t) \, dt \), we have

\[
G(-4) = \int_{0}^{-4} f(t) \, dt = -\int_{-4}^{0} f(t) \, dt = \left[ -2 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 \right] = 4.
\]

Finally, \( H(4) = \int_{-3}^{4} f(t) \, dt = -2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 1 + \frac{\pi}{2} \).

(b) Where is \( G \) increasing? Where is \( G \) decreasing?

For parts (b), (c), and (d), recall that we learned in class that \( G' = f \).

\( G \) is increasing where \( f \) is positive: \((-1, 4)\). Note that \( G \) has a horizontal slope at \( x = 2 \) but since \( f \) is positive on each side of \( t = 2 \), we say \( G \) is increasing at \( x = 2 \).

\( G \) is decreasing where \( f \) is negative: \((-4, -1)\).

(c) Where is \( G \) concave up? Where is \( G \) concave down?

\( G \) is concave up where \( f \) is increasing: \((-2, 0) \cup (2, 3)\).

\( G \) is concave down where \( f \) is decreasing: \((1, 2) \cup (3, 4)\).

(d) At what \( x \)-value(s) does \( G \) have a local maximum? At what \( x \)-value(s) does \( G \) have a local minimum?

\( G \) has a local maximum where \( f \) changes from positive to negative: never.

\( G \) has a local minimum where \( f \) changes from negative to positive: \( x = -1 \).

(e) Find a formula that relates \( G \) and \( H \).

From their definitions, \( H(x) = \int_{-3}^{x} f(t) \, dt + G(x) = -2 + G(x) \).

(f) How would your answers to (b), (c), and (d) change if the questions were about \( H \) instead of \( G \)?

They would not change at all because \( H'(x) = G'(x) \).

8. (a) Use sigma notation to express \( L_{10} \) and \( M_{10} \) as approximations to \( \int_{20}^{60} \ln x \, dx \).
We’re subdividing the interval into 10 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{10} = 4. \)

\[
L_{10} = [f(20) + f(24) + f(28) + \ldots + f(52) + f(56)]\Delta x
= [\ln(20) + \ln(24) + \ln(28) + \ldots + \ln(52) + \ln(56)] \cdot 4
= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4
\]

\[
M_{10} = [f(22) + f(26) + f(30) + \ldots + f(54) + f(58)]\Delta x
= [\ln(22) + \ln(26) + \ln(30) + \ldots + \ln(54) + \ln(58)] \cdot 4
= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4
\]

(b) **Draw a sketch that represents the sum \( M_{10}. \)**

Now we’re subdividing the interval into 4 pieces, so each piece has width \( \Delta x = \frac{60 - 20}{4} = 10. \)

Note that the height of each rectangle is determined by the \( y \)-value of the curve at the middle \( x \)-value of the rectangle (that is, at \( x = 25, 35, 45, 55 \)).

9. Find the following.

(a) **all antiderivatives of \( 1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5} \)**

Any such antiderivative will take the form \( x + x^2 + \frac{x^4}{4} + \frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + C. \)

Note that we have used the facts that \( \sqrt{x} = x^{1/2} \) and \( 1/x^5 = x^{-5} \).

(b) \[
\int_{-2}^{2} \sqrt{4 - x^2} \, dx = \frac{1}{2} \pi (2)^2 = 2\pi \quad \text{This integral represents the area of a semicircle of radius 2.}
\]

(c) \[
\frac{d}{dx} \int_{1}^{x} \sin \sqrt{t} \, dt = \sin \sqrt{x} \quad \text{The derivative of the area function is the original function.}
\]
(d) \( \int_0^2 x^2 \, dx \)

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \).

We will do this with a right-hand sum \( R_n \).

We subdivide \([0, 2]\) into \( n \) equal pieces, each of width \( \Delta x = \frac{2-0}{n} = \frac{2}{n} \).

Thus, \( x_1 = \frac{2}{n}, \ x_2 = \frac{4}{n}, \ x_3 = \frac{6}{n}, \ldots, \ x_n = \frac{2n}{n} \).

\[
\int_0^2 x^2 \, dx = \lim_{n \to \infty} R_n
\]

This is our limit definition of the definite integral.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k) \Delta x
\]

This is our definition of a right-hand sum.

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{2k}{n} \right)^2 \frac{2}{n}
\]

From above, \( x_k = \frac{2k}{n} \) and \( \Delta x = \frac{2}{n} \).

\[
= \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{4k^2}{n^2} \right) \frac{2}{n}
\]

Our function is \( f(x) = x^2 \).

\[
= \lim_{n \to \infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2
\]

We can pull out \( \frac{8}{n^3} \) because it doesn’t depend on \( k \).

\[
= \lim_{n \to \infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}
\]

We apply the handy fact we were given above.

\[
= \lim_{n \to \infty} \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \left( \frac{2n^2 + 3n + 1}{n^2} \right)
\]

\[
= \lim_{n \to \infty} \frac{4}{3} \left( \frac{2}{n} + \frac{3}{n^2} + \frac{1}{n^2} \right)
\]

\[
= \lim_{n \to \infty} \frac{4}{3} (2 + 0 + 0)
\]

\[
= \frac{8}{3}
\]

Now check with the FTC: \( \int_0^2 x^2 \, dx = \left[ \frac{x^3}{3} \right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3} \). That was slightly easier.