1. Consider the function \( f(x) = \frac{3}{5 - 2x} \).

(a) Is this function continuous on the interval \((-\infty, \infty)\)? Explain.

No. \( f \) is discontinuous at \( x = 2.5 \), where \( f \) is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of \( f \) on \([2, 2.01]\).

\[
\frac{f(2.01) - f(2)}{2.01 - 2} = \left[ \frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122
\]

(c) Using the limit definition of the derivative, compute \( f'(x) \).

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h}
\]

\[
= \lim_{h \to 0} \frac{3(5-2x) - 3(5-2(x+h))}{h[5-2(x+h)][5-2x]} = \lim_{h \to 0} \frac{15 - 6x - (15 - 6x - 6h)}{h[5-2(x+h)][5-2x]} = \lim_{h \to 0} \frac{6h}{6[5-2(x+h)][5-2x]} = \lim_{h \to 0} \frac{6}{(5 - 2x)^2}
\]

(d) Find the equation of the tangent line to \( f \) at \( x = 2 \).

We want \( y = mx + b \). \( m = f'(2) = \frac{6}{(5 - 2(2))^2} = 6 \), so \( y = 6x + b \).

[Note that this slope agrees well with our answer from (b) above.]

When \( x = 2 \), \( y = f(2) = \frac{3}{5 - 2(2)} = 3 \).

Thus, \( 3 = 6 \cdot 2 + b \), so \( b = -9 \) and we have \( y = 6x - 9 \).

2. Given that \( f(0) = 2 \), \( g(0) = 3 \), \( f'(0) = 5 \), \( g'(0) = 7 \), and \( f'(3) = \pi \) compute the following.

(a) \( h'(0) \) if \( h(x) = f(x)g(x) \)

\( h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29 \)

(b) \( j'(0) \) if \( j(x) = \frac{f(x)}{g(x)} \)

\( j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9} \)

(c) \( k'(0) \) if \( k(x) = f(g(x)) \)

\( k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi \)
3. Compute $dy/dx$ for each of the following.

(a) $y = x^5 + 5x + e^5 + \frac{x}{5} + \frac{5}{x} + \ln (5x) + \arctan (5x) + \ln(5) + \sin 5$

\[
\frac{dy}{dx} = 5x^4 + (\ln 5)5x + 1 + \frac{1}{5} - 5x^{-2} + 5 \cdot \frac{-1}{5} x^{-6/5} + \frac{1}{5x} \cdot 5 + \frac{1}{1 + (5x)^2} \cdot 5 + 0 + 0
\]

\[
= 5x^4 + (\ln 5)5x + \frac{1}{5} - \frac{1}{x^2} - \frac{1}{x} + \frac{5}{1 + 25x^2}
\]

(b) $y = \sqrt[3]{x} \cos(7x^3)$

\[
\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)
\]

(c) $y = \frac{e^x + e^y}{\tan 4 - 7x}$

\[
\frac{dy}{dx} = \frac{e^x(e^{\tan 4 - 7x}) - (-7)(e^x + e^y)}{(\tan 4 - 7x)^2}
\]

(d) $y = \arctan(e^{x^2} \arcsin(5x))$

\[
\frac{dy}{dx} = \sec^2(e^{x^2} \arcsin(5x)) \cdot e^{x^2} \arcsin(5x) \cdot \left[ x^2 \cdot \frac{1}{\sqrt{1 - 25x^2}} \cdot 5 + 2x \arcsin(5x) \right]
\]

(e) $y^3 + yx^2 + x^2 = 3y^2$

Here we use implicit differentiation.

\[
3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 + 2xy + 2x = 6y \frac{dy}{dx}
\]

\[
3y^2 \frac{dy}{dx} + \frac{dy}{dx} x^2 - 6y \frac{dy}{dx} = -2xy - 2x
\]

\[
\frac{dy}{dx}(3y^2 + x^2 - 6y) = -2xy - 2x
\]

\[
\frac{dy}{dx} = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}
\]

(f) $y = (x^2 + 1)^{\sin x}$ [Students in the 1:10 section may consider this as a bonus problem.]

Since we have $x$ in the base and the exponent, we need logarithmic differentiation.

\[
\ln y = \ln((x^2 + 1)^{\sin x})
\]

\[
\ln y = \sin x \cdot \ln(x^2 + 1)
\]

\[
\frac{1}{y} \frac{dy}{dx} = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x
\]

\[
\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot y
\]

\[
\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot (x^2 + 1)^{\sin x}
\]
4. Given the graph of \( f \), sketch a graph of \( f' \) and a graph of \( F \), an antiderivative of \( f \) such that \( F(0) = -1 \).

Note: The concave up portion on the left side of the graph of \( f \) is a perfect parabola, so its derivative \( (f') \) is linear; since you don’t know the equation for \( f \), your graph of \( f' \) may be concave up/down there.

5. Shown below is a graph of \( f' \) on its entire domain. The graph is NOT \( f \).

At which \( x \)-value(s)

(a) does \( f \) have a stationary point? \( c, f, h \)
(b) \( f \) decreasing? \( (c, f) \cup (h, j) \)
(c) \( f' \) increasing? \( [a, b) \cup (d, g) \cup (i, j] \)
(d) \( f' \) decreasing? \( (b, d) \cup (g, i) \)
(e) \( f \) concave up? \( [a, b) \cup (d, g) \cup (i, j] \)
(f) \( f \) concave down? \( (b, d) \cup (g, i) \)

(b) does \( f \) have a local max? \( c, h \)
(c) does \( f \) have a local min? \( f \)
(d) does \( f' \) have a stationary point? \( b, d, g, i \)
(e) does \( f' \) have a local max? \( b, g \)
(f) does \( f' \) have a local min? \( d, i \)
(g) is \( f \) greatest? \( c \)
(h) is \( f \) least? \( j \)
(i) is \( f' \) greatest? \( b \)
(j) is \( f' \) least? \( d \)
(k) is \( f'' \) greatest? \( e \)
(l) is \( f'' \) least? \( c \)

On what interval(s) is

(a) \( f \) increasing? \( [a, c) \cup (f, h) \)
6. Is \( y = 7e^{3x} \) a solution to the differential equation \( y'' + 2y' - 15y = 0 \)? Explain.

A given function \( y \) will be a solution to the differential equation if, when we substitute in \( y'', y' \), and \( y \), the equation is satisfied (that is, both sides of it are equal).

Since \( y = 7e^{3x} \), we know that \( y' = 21e^{3x} \) and \( y'' = 63e^{3x} \) from the Chain Rule.

Now we check to see whether our \( y \) satisfies the differential equation.

\[
y'' + 2y' - 15y = 63e^{3x} + 2 \cdot 21e^{3x} - 15 \cdot 7e^{3x} = 63e^{3x} + 42e^{3x} - 105e^{3x} = 0
\]

So, we see that \( y = 7e^{3x} \) is in fact a solution to this differential equation.

7. Rewrite \( \sin(\arctan(5x)) \) as an algebraic expression. [Students in the 8:00 section may omit this problem.]

Let \( \theta = \arctan(5x) \). That is, \( \theta \) is the angle whose tangent is \( 5x \).

We draw a triangle for which \( \frac{\text{opposite}}{\text{adjacent}} = \frac{5x}{1} = 5x \).

\[
\begin{align*}
\theta & \quad 1 \\
5x & \quad z \\
\end{align*}
\]

\[1^2 + (5x)^2 = z^2 \Rightarrow z = \sqrt{1 + 25x^2}\]

\[\sin(\arctan(5x)) = \sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{1 + 25x^2}}\]

8. Evaluate the following limits.

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L’Hopital’s Rule on the indeterminate form \( 0/0 \); this may be \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) or \( \frac{\infty}{\infty} \) or “has the form \( \frac{0}{0} \) and so, by L’Hopital’s Rule, is equal to” or something else. The symbol ♦ will serve the same purpose for the indeterminate form \( \infty/\infty \).

(a) \( \lim_{x \to \infty} \frac{x^2}{\ln x} \) ★ \( \lim_{x \to \infty} \frac{2x}{1/x} = \lim_{x \to \infty} 2x^2 = \infty \)

(b) \( \lim_{x \to 0} \frac{\sin(12x) - 12x}{x^3} \) ★ \( \lim_{x \to 0} \frac{12 \cos(12x) - 12}{3x^2} \) ★ \( \lim_{x \to 0} \frac{-144 \sin(12x)}{6x} \) ★ \( \lim_{x \to 0} \frac{-1728 \cos(12x)}{6} = -288 \)

(c) \( \lim_{x \to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0 \)

(d) \( \lim_{x \to 2} \frac{x^3 - 8}{x - 2} \) ★ \( \lim_{x \to 2} \frac{3x^2}{1} = 12 \)