Math 106: Review for Final Exam, Part I - SOLUTIONS

1. Find the following.  
   [See Review for Exam II for integration tips and strategies.]

   (a) Let \( u = x^3 \), so \( du = 3x^2 \, dx \) and \( du/3 = x^2 \, dx \).
   \[
   \int 12x^2 \cos(x^3) \, dx = 12 \int \cos(x^3) x^2 \, dx \\
   = 12 \int \cos(u) \frac{du}{3} \\
   = 4 \sin(u) + C \\
   = 4 \sin(x^3) + C
   \]

   (b) We’ll use integration by parts: \( u = x \Rightarrow du = dx \) and \( dv = e^{-3x} \Rightarrow v = \frac{e^{-3x}}{-3} \).
   \[
   \int_0^\infty xe^{-3x} \, dx = \lim_{t \to \infty} \int_0^t xe^{-3x} \, dx \\
   = \lim_{t \to \infty} \left[ \frac{x e^{-3x}}{-3} \bigg|_0^t - \int_0^t \frac{e^{-3x}}{-3} \, dx \right] \\
   = \lim_{t \to \infty} \left[ \frac{x e^{-3x}}{-3} - \frac{e^{-3x}}{-9} \right]_0^t \\
   = \lim_{t \to \infty} \left[ \frac{-t - \frac{1}{9} e^{3t}}{-3} - \frac{0 - \frac{1}{9} e^0}{-3} \right] \\
   = (0 - 0) - (0 - 1/9) \\
   = 1/9
   \]
   So, the integral converges (to this value).

   (c) This integral is improper at \( x = 4 \) because the integrand has a vertical asymptote there, so we split into two integrals.
   \[
   \int_0^6 \frac{dx}{(x-4)^2} = \int_0^4 \frac{dx}{(x-4)^2} + \int_4^6 \frac{dx}{(x-4)^2} \\
   = \lim_{a \to 4^-} \int_0^a \frac{dx}{(x-4)^2} + \lim_{b \to 4^+} \int_0^b \frac{dx}{(x-4)^2} \\
   = \lim_{a \to 4^-} \left[ \frac{-1}{x-4} \right]_0^a + \lim_{b \to 4^+} \left[ \frac{-1}{x-4} \right]_0^b \\
   = \lim_{a \to 4^-} \left[ \frac{-1}{(a-4)} - \frac{-1}{(0-4)} \right] + \lim_{b \to 4^+} \left[ \frac{-1}{(6-4)} - \frac{-1}{(b-4)} \right] \\
   \]
   Since \( \lim_{a \to 4^-} \frac{-1}{(a-4)} = \infty \) and \( \lim_{b \to 4^+} \frac{-1}{(b-4)} = \infty \), this integral diverges (to \( \infty \)).

   (d) Partial Fractions:
   Write \( \frac{3x^2 + 2x - 5}{(x^2+1)(x-4)} = \frac{Ax + B}{x^2+1} + \frac{C}{x-4} \). Now multiply both sides by \((x^2+1)(x-4)\) to get
   \[
   3x^2 + 2x - 5 = (Ax + B)(x-4) + C(x^2 + 1).
   \]
   Let \( x = 4 \). Then \( 51 = C(17) \), so \( C = 3 \).
   Let \( x = 0 \). Then \( -5 = B(-4) + 3(1) \), so \( B = 2 \).
Let \( x = 1 \). Then \( 0 = (A(1) + 2)(-3) + 3(2) \), so \( A = 0 \).

\[
\int \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} \, dx = \int \left[ \frac{2}{x^2 + 1} + \frac{3}{x - 4} \right] \, dx
\]

\[= 2 \arctan x + 3 \ln |x - 4| + D\]

(e) Let \( u = \sec x \), so \( du = \sec x \tan x \, dx \).

New limits: \( x = 0 \Rightarrow u = \sec 0 = 1 = 1/\cos 0 = 1 \) and \( x = \pi/3 \Rightarrow u = \sec(\pi/3) = 2/\cos(\pi/3) = 2 \).

\[
\int_{\pi/3}^{0} \tan^3 x \sec^5 x \, dx = \int_{0}^{\pi/3} \tan^3 x \sec^4 x \sec x \tan x \, dx
\]

Break off a \( \sec x \tan x \).

\[= \int_{0}^{\pi/3} (\sec^2 x - 1) \sec^4 x \sec x \tan x \, dx\]

Use \( \tan^2 x = \sec^2 x - 1 \).

\[= \int_{1}^{2} (u^2 - 1)u^4 \, du\]

Change the limits. See above.

\[= \int_{1}^{2} (u^6 - u^4) \, du\]

\[= \left[ \frac{u^7}{7} - \frac{u^5}{5} \right]_{1}^{2}\]

\[= \left[ \frac{2^7}{7} - \frac{2^5}{5} \right] - \left[ \frac{1^7}{7} - \frac{1^5}{5} \right]\]

\[= \frac{418}{35}\]

This is about 11.943.

(f) Let \( x = 5 \sin t \), so \( dx = 5 \cos t \, dt \).

\[x^2 + y^2 = 5^2 \Rightarrow y = \sqrt{25 - x^2}\]

\[\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5)\]

\[\cos t = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5 \cos t = \sqrt{25 - x^2}\]

\[
\int \sqrt{25 - x^2} \, dx = \int 5 \cos t \cdot 5 \cos t \, dt
\]

Use \( dx \) and \( \cos t \) from above.

\[= \int 25 \cos^2 t \, dt\]

Use \( \cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2} \) or table #42.

\[= 25 \int \left[ \frac{1}{2} + \frac{\cos(2t)}{2} \right] \, dt\]

Let \( u = 2t \) to integrate \( \cos(2t) \).

\[= 25 \left[ \frac{t}{2} + \frac{\sin(2t)}{4} \right] + C\]

Use \( \sin(2t) = 2 \sin t \cos t \) and \( x \) from above.

\[= 25 \left[ \arcsin(x/5) \cdot \frac{2}{2} + \frac{2 \sin t \cos t}{4} \right] + C\]

Use \( \sin t \) and \( \cos t \) from above.

\[= 25 \left[ \arcsin(x/5) \cdot \frac{2}{2} + \frac{2 \cdot \frac{x}{5} \cdot \sqrt{25 - x^2}}{4} \right] + C\]

\[= 25 \left[ \arcsin(x/5) \cdot \frac{2}{2} + \frac{x \sqrt{25 - x^2}}{50} \right] + C\]
2. Find the best possible left, right, midpoint, trapezoidal, and Simpson’s approximations to \( \int_{-2}^{0} f(x) \, dx \) given the data in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

\[ L_4 = (2 + 3 + 6 + 10)(0.5) = 10.5 \quad R_4 = (3 + 6 + 10 + 11)(0.5) = 15 \quad T_4 = 0.5(L_4 + R_4) = 12.75 \]

We cannot compute \( M_4 \), which would require the values of \( f \) at \( x = -1.75, -1.25, -0.75, \) and \(-0.25. \)

Instead, we find \( M_2 : M_2 = (3 + 10)(1) = 13 \).

Finally, \( S_4 = \frac{2M_2 + T_2}{3} \), so we need to compute \( T_2 = \frac{L_2 + R_2}{2} = \frac{(2 + 6)(1) + (6 + 11)(1)}{2} = 12.5 \).

Thus, \( S_4 = \frac{2M_2 + T_2}{3} = \frac{2(13) + 12.5}{3} = \frac{77}{6} \).

3. If you use numerical integration to estimate \( \int_{a}^{b} \ln x \, dx \) (where \( a \) and \( b \) are positive), how would the following be ordered from least to greatest? \( L_{100}, R_{100}, M_{100}, T_{100}, \int_{a}^{b} \ln x \, dx \).

The integrand is increasing and concave down, so we have \( L_{100} < T_{100} < \int_{a}^{b} \ln x \, dx < M_{100} < R_{100} \).

4. Find bounds for each of the following errors if \( I = \int_{0}^{2} e^{-5x} \, dx \).

\[ (a) \quad |I - R_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{5(2-0)^2}{2(100)} = \frac{1}{10} \]

\( K_1 = \text{max of } |f'(x)| \text{ on } [0, 2] = \text{max of } 5e^{-5x} \text{ on } [0, 2] = 5 \text{ (occurs at } x = 0) \)

\[ (b) \quad |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{25(2-0)^3}{12(100)^2} = \frac{1}{600} \]

\( K_2 = \text{max of } |f''(x)| \text{ on } [0, 2] = \text{max of } 25e^{-5x} \text{ on } [0, 2] = 25 \text{ (occurs at } x = 0) \)

\[ (c) \quad |I - M_{100}| \leq \frac{K_3(b-a)^3}{24n^2} = \frac{25(2-0)^3}{24(100)^2} = \frac{1}{1200} \]

\( K_3 = \text{same as in previous part} \)

5. If \( I = \int_{0}^{2} e^{-5x} \, dx \), how many subdivisions are required to obtain a midpoint sum approximation with error of at most \( 1/1,000,000? \)

From part (c) above, we know that \( |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{25(2-0)^3}{24n^2} = \frac{25}{3n^2} \)

Thus, we want \( \frac{25}{3n^2} \leq \frac{1}{1,000,000} \), which is equivalent to \( \frac{25,000,000}{3} \leq n^2 \).

Taking the square root of each side results in \( \sqrt{25,000,000}/3 \leq n \).

Since \( \sqrt{25,000,000}/3 = 2886.751... \), we must use at least 2887 subdivisions.
6. Write an integral equal to the area between \( y = 2x + 3 \) and \( y = x^2 + 7x - 3 \).

First, find where the curves intersect.

\[
\begin{align*}
x^2 + 7x - 3 &= 2x + 3 \\
x^2 + 5x - 6 &= 0 \\
(x + 6)(x - 1) &= 0 \\
\Rightarrow x &= -6, x = 1
\end{align*}
\]

Between \( x = -6 \) and \( x = 1 \), \( y = 2x + 3 \) is above \( y = x^2 + 7x - 3 \). (Plug in \( x = 0 \) to check.) So, the area between them is

\[
\int_{-6}^{1} [(2x + 3) - (x^2 + 7x - 3)] \, dx.
\]

[This equals 343/6.]

7. Compute the arc length of \( y = \sqrt{1 - x^2} \) from \( x = 0 \) to \( x = 1/2 \).

First, we find \( f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}} \).

Thus, \([f'(x)]^2 = \frac{x^2}{1 - x^2} \).

\[
\int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{1/2} \sqrt{1 + \frac{x^2}{1 - x^2}} \, dx
\]

This is the definition of arc length.

\[
= \int_{0}^{1/2} \sqrt{\frac{1 - x^2 + x^2}{1 - x^2}} \, dx
\]

Get a common denominator.

\[
= \int_{0}^{1/2} \frac{1}{\sqrt{1 - x^2}} \, dx
\]

\[
= \int_{0}^{1/2} \frac{\sqrt{1}}{\sqrt{1 - x^2}} \, dx
\]

\[
= \arcsin x \bigg|_{0}^{1/2}
\]

\[
= \arcsin(1/2) - \arcsin(0)
\]

\[
= \pi/6 - 0
\]

\[
= \pi/6
\]

8. Consider the region bounded by \( y = 0 \), \( x = 2 \), and \( y = x^2 \). Write an integral equal to the volume of the object created when the region is revolved about

(a) the \( x \)-axis

Slice vertically into disks.

\[
\text{volume of slice} \approx \pi r^2 \Delta x
\]

\[
= \pi y^2 \Delta x
\]

\[
= \pi (x^2)^2 \Delta x
\]

\[
= \pi x^4 \Delta x
\]

total volume = \( \pi \int_{0}^{2} x^4 \, dx \)

(b) the line \( x = 5 \)

Slice horizontally into washers.
The probability density function (pdf) of the weights of newborn toads in a certain pond is given by \( f(x) = \frac{k}{(x + 1)^4} \), where \( x \) is the weight (in ounces). Note that the domain is \( x \geq 0 \) since no toad can have a negative weight.

(a) **What must be the value of \( k \)?**

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

\[
\int_{0}^{\infty} \frac{k}{(x + 1)^4} \, dx = \lim_{t \to \infty} \int_{0}^{t} \frac{k}{(x + 1)^4} \, dx
\]

\[
= \lim_{t \to \infty} \frac{k(x + 1)^{-3}}{-3} \bigg|_{0}^{t}
\]

\[
= \lim_{t \to \infty} \frac{k}{-3(t + 1)^3} - \frac{k}{-3(0 + 1)^3}
\]

\[
= 0 - \frac{k}{-3}
\]

\[
= \frac{k}{3}
\]

So, we have \( k/3 = 1 \) or \( k = 3 \).

(b) **What fraction of the newborn toads weigh more than one ounce?**

\[
\int_{1}^{\infty} \frac{3}{(x + 1)^4} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{3}{(x + 1)^4} \, dx
\]

\[
= \lim_{t \to \infty} \frac{1}{-1(t + 1)^3} - \frac{1}{-1(1 + 1)^3}
\]

\[
= 0 - \frac{1}{-8}
\]

\[
= \frac{1}{8}
\]

Note that we could instead have computed \( 1 - \int_{0}^{1} \frac{3}{(x + 1)^4} \, dx \) and gotten the same answer.
10. Find the solution to \( \frac{dy}{dx} = \frac{\cos x}{y^2} \) that passes through \((0, 2)\).

We use separation of variables.

\[
\frac{dy}{dx} = \frac{\cos x}{y^2} \\
y^2 \, dy = \cos x \, dx \\
\int y^2 \, dy = \int \cos x \, dx \\
y^3/3 = \sin x + C \\
y^3 = 3 \sin x + D \\
y = \sqrt[3]{3 \sin x + D}
\]

When \(x = 0\), we have \(y = 2\), so \(2 = \sqrt[3]{3 \sin 0 + D}\), or \(2 = \sqrt[3]{D}\). Thus, \(D = 8\).

Therefore, the solution is \(y = \sqrt[3]{3 \sin x + 8}\).