Problem 1. (10 points) There is a mysterious function \( f \) which has the property that
\[
    f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 2, \quad \ldots \quad f^{(n)}(0) = n, \quad \ldots
\]

(a) (7 points) Determine a Maclaurin series for \( f \), and calculate its radius of convergence.

\[
    \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{(n-1)!} x^n
\]

Radius of convergence? Suspect \( \infty \) since \( \frac{1}{(n-1)!} \to 0 \) at factorial speed:

\[
    r_n = \frac{\frac{1}{(n-1)!} x^n}{\frac{1}{(n-2)!} x^{n-1}} = \frac{1}{n-1} x \quad \text{as} \quad n \to \infty \quad \Rightarrow \quad 0 \quad \text{and} \quad |x| < 1 \quad \text{regardless of} \quad x.
\]

Thus radius is infinite: the Maclaurin series converges for all real \( x \).

(b) (3 points) What function does this Maclaurin series represent?

Hint: divide every term by \( x \), and you should get something familiar.

\[
    \sum_{n=0}^{\infty} \frac{1}{(n-1)!} \left( \frac{x}{n} \right)^n = x + x^2 + \frac{1}{2!} x^3 + \frac{1}{3!} x^4 + \ldots
\]

\[
    \times \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{x}{n} \right)^{n-1} = x \left[ 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots \right]
\]

\[
    \times \sum_{n=0}^{\infty} \frac{1}{n!} x^n = x \left[ 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots + \frac{1}{n!} x^n + \ldots \right]
\]

"Usual suspect" = \( xe^x \)
Problem 2. (10 points) The series $\sum_{n=0}^{\infty} \frac{n(-1)^n}{(n+1)(n+2)}$ is conditionally convergent. Use this fact to give an example of a power series whose interval of convergence is exactly $4 < x \leq 10$.

Hint: tweak the asymptotic ratio of this coefficient series first.

Interval $4 < x \leq 10$ has radius $\frac{10-4}{2} = 3$, and center point $\frac{10+4}{2} = 7$.

Thus need coefficient series with asympt. ratio $= \frac{1}{3}$.

Since our given series is polynomial, it has $r = 1$. Divide it by $3^n$ to get

$$\sum_{n=0}^{\infty} \frac{n(-1)^n}{(n+1)(n+2)} 3^n$$

with asymptotic ratio $\frac{1}{3}$.

Build a power series with this coeff. series centered at $x = 7$:

$$\sum_{n=0}^{\infty} \frac{n(-1)^n}{(n+1)(n+2)} 3^n (x-7)^n$$

It converges at least on $(4, 10)$.

At $x = 10$

$$\sum_{n=0}^{\infty} \frac{n(-1)^n}{(n+1)(n+2)}$$

we’re told this is conditionally convergent

$$\sum_{n=0}^{\infty} \frac{n}{(n+1)(n+2)}$$

diverges.

Thus $(4, 10]$ is its interval.

Problem 3. (5 points) Shown at left is a graph of the function $g(x) = \frac{1}{5-x}$.

If you were to construct a Taylor series for $g(x)$ with base point $a = 2$, what would you expect its interval of convergence to be? Why?

Hint: don’t waste any time trying to calculate this Taylor series.

It cannot converge on any interval big enough to include $x = 5$.

But if we cannot include 5, we cannot include anything else $\geq 3$ units away from center either.

Expect radius of convergence = 3 and interval of convergence

$$-1 < x < 5 \text{ or } -1 \leq x \leq 5.$$