1. Let \( A = \begin{bmatrix} 6 & 1 & 0 \\ 12 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \).

1A. Find the characteristic polynomial of \( A \) in factored form. Show all your work.

That polynomial is 
\[
\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & 1 & 0 \\ 12 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 6-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}
\]

\[
= (2-\lambda)((6-\lambda)(2-\lambda) - 12)
\]

\[
= (2-\lambda)(30 - 11\lambda + \lambda^2 - 12)
\]

\[
= (2-\lambda)(\lambda^2 - 11\lambda + 18)
\]

\[
= (2-\lambda)(\lambda - 9)(\lambda - 2)
\]

or \((-1)(\lambda - 2)(\lambda - 9)\lambda - q\)

1B. Use (1A) to list all the eigenvalues of \( A \) along with their respective multiplicities:

So: \( \lambda = 2 \) with multiplicity 2 and \( \lambda = 9 \) with multiplicity 1.

2. Let \( B = \begin{bmatrix} 6 & -4 & 2 \\ 1 & 2 & 1 \\ 1 & -2 & 5 \end{bmatrix} \). It's a fact (you don't need to show this) that \( \lambda = 4 \) is an eigenvalue of \( B \) of multiplicity 2.

2A. Find a basis of the eigenspace corresponding to \( \lambda = 4 \). Show all your work, starting by explicitly finding \( B - 4I_3 \).

This is the same question as "Find a basis of the null space of \( B - 4I \"

Now, \( B - 4I = \begin{bmatrix} 2 & -4 & 2 \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \) has ref \( \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \), so the solutions of \( (B - 4I)x = 0 \) are given by

\[
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}
\]

where \( x_2 \) and \( x_3 \) are free. A basis is then: \( \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \)

2B. Find \( Bc \), where \( c = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \). Use a calculator to get \( \begin{bmatrix} 12 \\ 16 \\ 20 \end{bmatrix} \).

2C. Show that \( c = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \) is in the eigenspace of \( \lambda = 4 \) by writing \( c \) as a linear combination of the basis vector(s) you found in (2A). (Give the LC explicitly).

That easy-to-use basis gives

\[
4 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}
\]

(although you should "verify" the top row? yes, at right: \( 4 \cdot 2 + 5 \cdot -1 = 3 \).)

2D. Compute \( Bk \) for \( k = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \). What do you discover from this?

\( Bk \) is \( \begin{bmatrix} 10 \\ 5 \\ 5 \end{bmatrix} \) which is \( 5 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \) showing \( \lambda = 5 \) is another eigenvalue of \( B \).

2E. Using the discovery in (2D) and the fact given at the start of problem (2), find the characteristic polynomial of \( B \) in factored form (no need to multiply it all out).

Since \( \lambda = 4 \) has multi. 2 we know \((\lambda - 4)^2 \) is one factor.

Since the degree of the poly is 3 and \( \lambda = 5 \) is also an eigen vector, \((\lambda - 5) \) must be the other factor

So char poly \( B \) is \((\lambda - 4)^1(\lambda - 5)^{2} \)