For each of the infinite series in #1—5, (a) state your intuition on whether or not it converges conditionally, converges absolutely, or diverges, and (b) prove your assertion with your choice of convergence test(s).

**Problem 1.** (5 points) \[ \sum_{n=0}^{\infty} \frac{79^n}{n!} \]

**a.** Dominated by factorial behavior. Expect asymptotic ratio = 0 and hence convergence.

**b.** \[ r_n = \frac{a_n}{a_{n-1}} = \frac{\frac{79^n}{n!}}{\frac{79^{n-1}}{(n-1)!}} = \frac{79}{n} \xrightarrow{n \to \infty} 0. \]

Since \( r_n = 0 < 1 \), the ratio test implies this asymptotically geometric series converges (Absolutely, since all terms are already positive.)

**Problem 2.** (5 points) \[ \sum_{n=1}^{\infty} \frac{(-1/2)^n}{\sqrt{n} + 4} \]

**a.** Dominated by geometric behavior, even after alternation is discarded.

Expect convergence, since the ratios \((\frac{1}{2})\) are < 1.

**b.** \( \lim |a_n| = \lim \left| \frac{(-1)^n}{2^n \sqrt{n} + 4} \right| = \lim \frac{1}{2^n \sqrt{n+4}} \sim \frac{1}{\infty} = 0 \)

Since \( |a_n| \to 0 \) we know \( a_n \to 0 \) and this converges by the alternating series test!

Moreover, the series \( \sum_{n=1}^{\infty} 2^n \sqrt{n+4} \) has ratios \( R_n = \frac{2^{n-1} \sqrt{n+3}}{2^n \sqrt{n+4}} = \frac{1}{2} \sqrt{\frac{n+3}{n+4}} \xrightarrow{n \to \infty} \frac{1}{2} < 1 \) giving absolute convergence by the ratio test.
Problem 3. (5 points) \( \sum_{n=1}^{\infty} \left( \frac{n}{3n+1} \right)^n \)

(a) Behavior looks like competing factorials. As \( n \to \infty \) this approximately looks like the geometric series \( \sum (\frac{1}{3})^n \) which converges.

(b) Use the root characterization for asymptotic ratios:

\[
R_n = \sqrt[n]{a_n} = \sqrt[n]{\left( \frac{n}{3n+1} \right)^n} = \frac{n}{3n+1} \quad \frac{n \to \infty}{\rightarrow} \quad \frac{1}{3} < 1,
\]

giving absolute convergence by the ratio test.

Problem 4. (5 points) \( \sum_{n=1}^{\infty} \frac{1}{16 + n^2} \)

(a) Were it not for the 16, would look like \( \sum \frac{1}{n^2} \) which converges as a "fast polynomial" series.

(b) The integral test casts this series as a left sum for the integral

\[
\int_{0}^{\infty} \frac{1}{16 + x^2} \, dx.
\]

Since \( \frac{1}{16 + x^2} \) is decreasing on \([0, \infty)\), the integral exceeds its left sum.

But 
\[
\int_{0}^{\infty} \frac{1}{16 + x^2} \, dx = \frac{1}{4} \arctan \frac{x}{4} \bigg|_{0}^{\infty} = \frac{\pi}{8},
\]

so in particular the integral is finite.

Thus \( \sum_{n=1}^{\infty} \frac{1}{16n^2} \) converges (to a value no bigger than \( \frac{\pi}{8} \)).

Problem 5. (5 points) \( \frac{2}{2} - \frac{4}{4} + \frac{8}{6} - \frac{16}{8} + \frac{32}{10} - \frac{64}{12} + \cdots \)

\[= \sum_{n=1}^{\infty} \frac{2^n}{2n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{n} \]

(a) It appears the numerator, being geometric, will set the pace over the polynomial denominator. But the numerator \( \to \infty \), so the terms \( \to \infty \). Diverges.

(b) With \( \ell ^{th} \) partial:

\[
\lim_{n \to \infty} \frac{2^{n-1}}{n} = \lim_{n \to \infty} \frac{(\ln 2)2^{n-1}}{n} = \frac{\infty}{\infty} = \infty
\]

with asymptotic ratios:

\[
r_n = \frac{2^{n-1}/n}{2^n/n} = \frac{2}{2} \, \frac{n-1}{n} \quad \frac{n \to \infty}{\rightarrow} \quad 2 > 1
\]

Diverges by ratio test.